ELEC547 Project Report

Application on Design of Multiuser Non-Cooperative Communication Systems via Game Theory

Zhen Wang

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Abstract

This project considers the application of convex optimization on design of multiuser non-cooperative communication systems via game theory. The interference channel is modeled as a non-cooperative power competing game and its Nash Equilibrium (NE) is the solution. There are three questions to ask: 1) whether the NE exists? 2) if it exists, whether it is unique? 3) if it is unique, how to find the equilibrium point? This problem has been studied in a number of papers. A variety of conditions guaranteeing the uniqueness of the NE and convergence of many different distributed algorithms have been derived. This project is part of my Final Year Thesis project and provides a survey on the first two questions, i.e. existence and uniqueness of the NE points. The last question about the convergence of the distributed algorithms will be studied afterwards.

I Introduction

In a general noncooperative game, there are N players, each of whom has a certain cost function and strategy set that may depend on the other players' actions. Assume that player i's strategy set is $\mathcal{K}_i$, which is a subset of $\mathcal{R}^{n_i}$ and is independent of the other players' actions. Player i's cost function $\theta_i(\mathbf{x})$ depends on all players' strategies, which are described by the vector $\mathbf{x}$ that consists of the subvectors $x^i \in \mathcal{R}^{n_i}$ for $i = 1, ..., N$. Player i's problem is to determine, for each fixed but arbitrary tuple $\tilde{\mathbf{x}}^i = (x^j : j \neq i)$ of other players' strategies, an optimal strategy x^i that solves the cost minimization problem in the variable x^i:

$$\begin{array}{ll} \text{minimize}_{\{x^i\}} & \theta_i(x^i, \tilde{\mathbf{x}}^i) \\ \text{subject to} & x^i \in \mathcal{K}_i \end{array} \tag{1}$$

The solution set of the optimization problem is denoted by $S_i(\tilde{\mathbf{x}}^i)$. A Nash Equilibrium (NE) is a tuple of strategies $\mathbf{x} = (x^i : i = 1, ..., N)$ with the property that for each i, $x^i \in S_i(\tilde{\mathbf{x}}^i)$. In words, a NE is a tuple of strategies, one for each player, such that no player can lower the cost by unilaterally deviating his action from his designated strategy.

The interference channel is a mathematical model relevant to many physical communication channels and multiuser systems where multiple uncoordinated links share a common communication medium. It is characterized by its capacity region, defined as the set of rates that can be simultaneously achieved by the users in the system. In principle, this multi-objective optimization of the communication system requires a centralized solution, which has high complexity, heavy signaling and the coordination among the users. Instead, the system can be designed in a game theoretical approach in a fully distributed fashion with no centralized control. The basic idea is that users in the system can be modeled as several players having different objectives, say maximizing their own mutual information or data rate, and behave selfishly without cooperation. Therefore, the original multi-objective optimization problem is converted into a set of mutually coupled single-objective optimization problems. The optimal solution to this approach is the achievement of NE. The existence and uniqueness of the NE have been studied in a number of works together with several iterative and distributed algorithms that converge to NE points. This project mainly focuses on the proof of the existence and uniqueness of NE of the strategic non-cooperative game, which is part of my Final Year Thesis project. The convergence of the distributed algorithms will be studied afterwards.

Eight papers [1, 2, 3, 4, 5, 6, 7, 8] are investigated in this report. [1, 2, 3] considers the two-user single-input single-output system and they are the seminal papers studying the strategic non-cooperative power competing game. [4, 5, 6, 7, 8] extend the problem to an arbitrary number of users. Among those papers, [1, 2, 3, 4, 7] considers the maximization of the data rate; [5, 6] considers the maximization of the mutual information; and [8] considers both cases and proves that they are equivalent and have a unified formulation. As for the proof of the existence and uniqueness of the NE points, this report focus on the proofs from [7] and [8]. While [7] is based on a key result that establishes a reformulation of the noncooperative Nash game as a linear complementarity problem (LCP), [8] applies problem simplification and the result for K-matrix.

This report is organized as follows. In Section II, general system model and problem formulation are described. Section III shows the proof based on majorization theory [9] from [8] about the equivalence of two maximization approaches, i.e. maximization of mutual information and maximization of data rate. Section IV give the proof of the existence and uniqueness of the NE points from two different methods in [7] and [8], respectively, i.e. by LCP formulation and by problem simplification and K-matrix formulation. Section V compares the results from those eight papers. Section VI gives some concluding remarks.

II General System Model and Problem Formulation

In this section, a general system model is provided and the problem is formulated by game theoretical approach. The constraints are also clarified.

A System Model

A general system model is represented by

$$y_q = H_{qq}x_q + \sum_{r\neq q} H_{rq}x_q + n_q, \tag{2}$$

where x_q is the vector transmitted by source q; y_q is the vector received by destination q; H_{qq} is the direct channel of link q; H_{rq} is the cross-channel matrix between source r and destination q; and n_q is the noise vector. $\sum_{r\neq q} H_{rq}x_q$ represents the multi-user interference (MUI) received by the qth destination and caused by other links; and it can be treated as additive noise.

This system model is sufficiently general to represent many cases of practical interest, such as digital subsciber line, cellular radio and ad hoc wireless networks. It assumes that each destination has perfect knowledge of the channel from its source, but not of the channel from the interfering sources.

B Problem Formulation

Using game theoretical approach, each user competes against the others in this strategic non-cooperative power competing game by choosing his power allocation (i.e. strategy) to maximize his mutual information or data rate (i.e. payoff). Reformulating the system within the framework of game theory, the strategic non-cooperative game has the following structure:

$$\begin{array}{lll} (\mathcal{G}) & \text{maximize}_{\{\mathbf{p}_q\}} & R_q(\mathbf{p}_q, \mathbf{p}_{-q}) \\ & \text{subject to} & \mathbf{p}_q \in \mathcal{P}_q \\ & & \forall q \in \mathcal{Q} \end{array} \tag{3}$$

where Q is the set of players; P_q is the set of admissible strategies for player q; $R_q(\mathbf{p}_q, \mathbf{p}_{-q})$ is the payoff function of player q. To solve the problem, the existence and uniqueness of the solution, i.e. NE, are needed to be considered.

The payoff function $R_q(\mathbf{p}_q, \mathbf{p}_{-q})$ can be different for different practical interest. Maximization of mutual information and maximization of data rate are two major payoff functions considered in the strategic non-cooperative power competing game.Those two payoff function are actually equivalent and the proof will be provided in the next section.

The physical constraints P_q required by the application are: 1) maximum transmit power for each user and 2) spectral mask constraints. While [1, 2, 3, 4, 5, 6] only consider the first constraint, [7, 8] consider both constraints.

Specific problem formulation will be provided when discussing the details of the proofs.

III Equivalence of Two Maximization Approaches [8]

The proof from [8] for the equivalence of two maximization approaches is shown in this section. The system model is described by (2) and $x_q = F_q s_q$ where s_q is the information symbol vector and F_q is the precoding vector. A cyclic prefix is incorporated on each transmitted block x_q so that H_{rq} can be diagonalized as

$$H_{rq} = W D_{rq} W^H \tag{4}$$

The physical constraints are as follows:

Co1) Maximum transmit power for each transmitter, i.e.,

$$E\{\|x_q\|_2^2\} = \frac{1}{N} Tr(F_q F_q^H) \leq p_q \tag{5}$$

Co2) Spectral mask constraint, i.e.,

$$E\{|[W^H F_q s_q]_k|^2\} = [W^H F_q F_Q^H W]_{kk} \leq \bar{p}_q^{max}(k) \tag{6}$$

A Problem Formulation for the two approaches

A.1 Maximization of Mutual Information

The mutual information for the qth user is:

$$I_q(F_q, F_{-q}) = \frac{1}{N} log(|I + F_q^H H_{qq}^H R_{-q}^{-1} H_{qq} F_q|) \tag{7}$$

where

$$R_{-q} = \sigma_q^2 I + \sum_{r \neq q} H_{rq} F_r F_r^H H_{rq}^H \tag{8}$$

is the interference-plus-noise covariance matrix observed by user q. Then we have the following strategic noncooperative game:

$$\begin{array}{lll} (\mathcal{G}_1) & \text{maximize}_{\{F_q\}} & I_q(F_q, F_{-q}) \\ & \text{subject to} & F_q \in \mathcal{F}_q \\ & & \forall q \in \mathcal{Q} \end{array} \tag{9}$$

where $I_q(F_q, F_{-q})$ is the payoff function defined in (7), $\mathcal{Q}$ is the set of players, and $\mathcal{F}_q$ is the admissible strategies of player q, defined as

$$\mathcal{F}_q = \{F_q \in \mathcal{C}^{N\times N} : \frac{1}{N} Tr(F_q F_q^H) \leq p_q, [W^H F_q F_q^H W]_{kk} \leq \bar{p}_q^{max}(k), \forall k \in 1, ..., N\} \tag{10}$$

A.2 Maximization of data rate

The transmission rate of each link is

$$r_q(F_q, F_{-q}) = \frac{1}{N}\sum_{k=1}^{N} log_2(1 + \frac{SINR_{k,q}(F_q, F_{-q})}{\Gamma_q}) \tag{11}$$

where

$$SINR_{k,q}(F_q, F_{-q}) = \frac{1}{[(I + F_q^H H_{qq}^H R_{-q}^{-1} H_{qq} F_q)^{-1}]_{kk}} - 1 \tag{12}$$

and $\Gamma_q \geq 1$ is the gap that depends only on the constellations and on the error probability. So the structure of the game is

$$\begin{array}{lll} (\mathcal{G}_2) & \text{maximize}_{\{F_q\}} & r_q(F_q, F_{-q}) \\ & \text{subject to} & F_q \in \mathcal{F}_q \\ & & \forall q \in \mathcal{Q} \end{array} \tag{13}$$

where $r_q(F_q, F_{-q})$ is defined in (11), $\mathcal{F}_q$ is defined in (10), and $\mathcal{Q}$ is the set of players.

It is proved in [8] that every NE of the game is achieved using pure strategy, so only pure strategies are considered here. The proof will not be discussed in this report.

B Theorem and the Proof

Theorem 1: An optimal solution to the matrix-valued games $\mathcal{G}_1$ and $\mathcal{G}_2$ is

$$F_q = W\sqrt{diag(\mathbf{p}_q)}, \forall q \in \mathcal{Q} \tag{14}$$

where W is the IFFT matrix in (4) and $\mathbf{p}_q = (p_q(k))_{k=1}^N$ is the solution to the vectore-valued game $\mathcal{G}$, defined in (3), where

$$R_q(\mathbf{p}_q, \mathbf{p}_{-q}) = \frac{1}{N}\sum_{k=1}^{N} log(1 + \frac{1}{\Gamma_q} sinr_q(k)) \tag{15}$$

with $sinr_q(k) = \frac{|H_{qq}(k)|^2 p_q(k)}{1+\sum_{r\neq q}|H_{rq}(k)|^2 p_r(k)}$ and

$$\mathcal{P}_q = \{\mathbf{p}_q \in \mathcal{R}^N : \frac{1}{N}\sum_{k=1}^{N} p_q(k) \leq 1, 0 \leq p_q(k) \leq p_q^{max}(k), \forall k \in 1, ..., N\} \tag{16}$$

with $p_q^{max}(k) = \bar{p}_q^{max}(k)/p_q$

Proof: We first prove *Theorem 1* for game $\mathcal{G}_1$. Then we show that the same result holds true for game $\mathcal{G}_2$.

Game $\mathcal{G}_1$:

Assume the optimal strategies of the other players are $F_r = W\Sigma_r^{1/2}$, with $\Sigma_r = diag(\mathbf{p}_r), \forall r \neq q$. From (8), we have

$$\begin{aligned} H_{qq}^H R_{-q}^{-1} H_{qq} &= H_{qq}^H(\sigma_q^2 I + \sum_{r\neq q} H_{rq} F_r F_r^H H_{rq}^H)^{-1} H_{qq} \\ &= H_{qq}^H(\sigma_q^2 I + \sum_{r\neq q} H_{rq} W \Sigma_r W^H H_{rq}^H)^{-1} H_{qq} \\ &= W D_{qq}^H W^H(\sigma_q^2 I + \sum_{r\neq q} W D_{rq} \Sigma_r D_{rq}^H W^H)^{-1} W D_{qq} W^H \\ &= W \Lambda_q W^H \end{aligned} \tag{17}$$

with Λ_q diagonal and $[\Lambda_q]_{kk} = \frac{|H_{qq}(k)|^2}{1+\sum_{r\neq q}|H_{rq}(k)|^2 p_r(k)}$. Defining $Q_q = W^H F_q F_q^H W$, from (7), we have

$$
\begin{aligned}
I_q(F_q, F_{-q}) &= \frac{1}{N} log(|I + F_q^H H_{qq}^H R_{-q}^{-1} H_{qq} F_q|) \\
&= \frac{1}{N} log(|I + F_q^H W \Lambda_q W^H F_q|) \\
&= \frac{1}{N} log(|I + \Lambda_q W^H F_q F_q^H W|) \\
&= \frac{1}{N} log(|I + \Lambda_q Q_q|) \\
&\leq \frac{1}{N} \sum_k log(1 + [\Lambda_q]_{kk}[Q_q]_{kk}) \text{ (Hadamard's inequality)} \qquad (18)
\end{aligned}
$$

with equality reached if and only if Q_q is diagonal. And since the power constraint (5) $Tr(Q_q) = Tr(W^H F_q F_q^H W) = Tr(F_q F_q^H) \leq N p_q$ and spectral mask constraint (6) $[W^H F_q F_q^H W]_{kk} = [Q_q]_{kk} \leq \bar{p}_q^{max}(k)$ depend only on the diagonal elements of Q_q, we may set Q_q diagonal, i.e. $Q_q = \Sigma_q$, which leads to the desired optimal structure for $F_q = W\Sigma_q^{1/2} = W\sqrt{diag(\mathbf{p}_q)}$. Introducing this optimal structure in $\mathcal{G}_1$, we obtain the simpler game $\mathcal{G}$ in (3) with $\Gamma_q = 1$.

Game $\mathcal{G}_2$:

This proof is based on majorization theory [9] and the key definitions and results are outlined as follows.

Definition 1: For any two vectors $x, y \in \mathcal{R}^n$, we say x is majorized by y or y majorizes x (denoted by $x \prec y$ or $y \succ x$) if

$$
\begin{aligned}
\sum_{k=1}^{i} x_{[k]} &\leq \sum_{k=1}^{i} y_{[k]}, 1 \leq i \leq n \\
\sum_{k=1}^{n} x_{[k]} &= \sum_{k=1}^{n} y_{[k]}
\end{aligned}
\qquad (19)
$$

where $x_{[k]}$ and $y_{[k]}$ denote the elements of x and y, respectively, in decreasing order.

Definition 2: A real valued function ϕ defined on a set $\mathcal{A} \subseteq \mathcal{R}^n$ is said to be Schur convex on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y) \qquad (20)$$

or Schur concave on $\mathcal{A}$ if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \geq \phi(y) \qquad (21)$$

.

Lemma 1: For a Hermitian matrix A and a unitary matrix U, it follows that

$$1(A) \prec d(U^H A U) \prec \lambda(A) \qquad (22)$$

. where $d(A)$ and $\lambda(A)$ denote the diagonal elements and eigenvalues of A, respectively, and $1(A)$ denotes the vector with identical components equal to the average of the diagonal elements of A.

Assume the optimal strategies of the other players are $F_r = W\Sigma_r^{1/2}$, with $\Sigma_r = diag(\mathbf{p}_r), \forall r \neq q$. Defining $P_q = W^H F_q$, the MSE matrix of user q can be written as

$$
\begin{aligned}
E_q(P_q) &= (I + F_q^H H_{qq}^H R_{-q}^{-1} H_{qq} F_q)^{-1})^{-1} \\
&= (I + F_q^H W \Lambda_q W^H F_q)^{-1})^{-1} \text{ (from (17))} \\
&= (I + P_q^H \Lambda_q P_q)^{-1} \qquad (23)
\end{aligned}
$$

$E_q(P_q)$ has two properties: i) it is a continuous function of $P_q \in \mathcal{C}^{N\times N}$ and ii) for any unitary matrix U, it satisfies $E_q(P_q U) = U^H E_P(P_q) U$.

Since Game $\mathcal{G}_2$ minimizes $r_q(F_q, F_{-q})$, from (11), we can write:

$$f_q(d(E_q(P_q))) = -r_q(F_q) = -\frac{1}{N}\sum_{k=1}^{N} log_2(1 + \frac{([E_q(P_q)]_{kk})^{-1} - 1}{\Gamma_q}). \tag{24}$$

$f_q(d(E_q(P_q)))$ has two properties: i) it is a continuous function of $P_q \in \mathcal{C}^{N\times N}$ and ii) it is Schur-concave on $\mathcal{R}_+^N$ [10].

Using (23) and (24), game $\mathcal{G}_2$ becomes

$$\begin{aligned} (P1) \quad \text{maximize}_{\{P_q\}} \quad & f_q(d(E_q(P_q))) \\ \text{subject to} \quad & \frac{1}{N} Tr(P_q P_q^H) \le p_q \\ & d(P_q P_q^H) \le \bar{p}_q^{max} = \bar{p}_q^{max}(k))_{k=1}^N \end{aligned} \tag{25}$$

Since the feasible set is closed and bounded, thus compact, and the objective function is continuous in its interior, problem P1 always admits a solution. Assume the solution is P_q^*. As $d(E_q(P_q))$ and $Tr(E_q(P_q))$ depend on $d(E_q(P_q))$ only, problem P1 is equivalent to

$$\begin{aligned} (P2) \quad \text{maximize}_{\{P_q\}} \quad & f_q(d(E_q(P_q))) \\ \text{subject to} \quad & d(P_q P_q^H) = d_q^* = d(P_q^* P_q^{*H}) \end{aligned} \tag{26}$$

Defining $\tilde{P}_q = P_q U$, the constraint is not affected: $d(\tilde{P}_q \tilde{P}_q^H) = d(P_q U U^H P_q^H) = d(P_q P_q^H)$. Since $f_q(d(E_q(P_q)))$ is Schur-concave, we have

$$d(E_q(P_q)) \prec \lambda(E_q(P_q)) \quad \Rightarrow \quad f_q(d(E_q(P_q))) \ge f_q(\lambda(E_q(P_q))) \tag{27}$$

$$d(E_q(\tilde{P}_q)) = d(U^H E_q(P_q) U) \prec \lambda(E_q(P_q)) \quad \Rightarrow \quad f_q(d(E_q(\tilde{P}_q))) \ge f_q(\lambda(E_q(P_q))) \tag{28}$$

When U diagonalize $E_q(P_q)$, $d(E_q(\tilde{P}_q)) = d(U^H E_q(P_q) U) = \lambda(E_q(P_q))$; and when $E_q(P_q)$ is diagonal, $d(E_q(P_q)) = d(E_q(\tilde{P}_q)) = \lambda(E_q(P_q))$; thus $f_q(d(E_q(P_q))) = f_q(\lambda(E_q(P_q)))$ is minimized. So $E_q(P_q)$ is diagonal with an optimal P_q, i.e. $P_q^H \Lambda_q P_q$ is diagonal from (23). Define $P_q^H \Lambda_q P_q = \Sigma_q$, i.e. $P_q = \Lambda_q^{-1/2} U_q \Sigma_q^{-1/2}$ and $d(P_q P_q^H) = d(U_q \Sigma_q U_q^H) = \bar{d}_q^* = \Lambda_q d_q^*$, where U_q is any unitary matrix. From (24), we have

$$\begin{aligned} f_q(d(E_q(P_q))) &= -\frac{1}{N}\sum_{k=1}^{N} log_2(1 + \frac{([(I+\Sigma_q)^{-1}]_{kk})^{-1} - 1}{\Gamma_q}) \\ &= -\frac{1}{N}\sum_{k=1}^{N} log_2(1 + \frac{[\Sigma_q]_{kk}}{\Gamma_q}) \end{aligned} \tag{29}$$

So problem P1 is equivalent to

$$\begin{aligned} (P3) \quad \text{maximize}_{\{\Sigma_q, U_q\}} \quad & -\frac{1}{N}\sum_{k=1}^{N} log_2(1 + \frac{[\Sigma_q]_{kk}}{\Gamma_q}) \\ \text{subject to} \quad & d(U_q \Sigma_q U_q^H) = \bar{d}_q^* \end{aligned} \tag{30}$$

Since we can always find a U_q satisfying the optimal solution Σ_q^* and $\bar{d}_q^* = d(U_q \Sigma_q U_q^H) \prec \lambda(\Sigma_q) = d(\Sigma_q)$, we have

$$\begin{aligned} (P4) \quad \text{maximize}_{\{\Sigma_q\}} \quad & -\frac{1}{N}\sum_{k=1}^{N} log_2(1 + \frac{[\Sigma_q]_{kk}}{\Gamma_q}) \\ \text{subject to} \quad & d(\Sigma_q) \succ \bar{d}_q^* \end{aligned} \tag{31}$$

Since the objective function is Schur-convex, the optimal solution Σ_q^* satisfies $d(\Sigma_q^*) = \bar{d}_q^*$. As we need U_q satisfying $d(U_q \Sigma_Q^* U_q^H) = \bar{d}_q^* = d(\Sigma_q^*)$, we get $U_q^* = I$. Therefore, $P_q^* = \Lambda_q^{-1/2} \Sigma_q^{*1/2}$, which leads to the desired expression $F_q^* = W P_q^* = W\sqrt{diag(\mathbf{p}_q)}$. This completes the proof.

IV Existence and Uniqueness of NE

This section investigates the existence and uniqueness of the NE from two different methods in [8] and [7], respectively. While [8] uses problem simplification and the result of K-matrix, [7] reformulates the strategic noncooperative power competing game by LCP and applies the result of LCPs. The proof in [8] will be discussed first, then the proof in [7].

A Proof by Problem Simplification and K-matrix Formulation [8]

Given game $\mathcal{G}$, define the nonnegative matrix $S(k) \in \mathcal{R}_{+}^{\mathcal{Q}\times\mathcal{Q}}$ as

$$[S(k)]_{qr} = \begin{cases} \Gamma_q \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2}, & \text{if } k \in \mathcal{D}_q \cap \mathcal{D}_r, r \neq q, \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

where $\mathcal{D}_q$ is any subset of $\{1,...,N\}$ such that $\mathcal{D}_q^{max} \subseteq \mathcal{D}_q \subseteq \{1,...,N\}$, with $\mathcal{D}_q^{min}$ denoting the set $\{1,...,N\}$ deprived of the carrier indexes that user q would never use as the best response set to any strategy used by the other users.

Theorem 2: Game $\mathcal{G}$ admits a nonempty solution set for any set of channels, transmit power and spectral mask constraints of the user. Furthermore, the NE is unique if

$$\rho(S(k)) < 1, \forall k \in \{1,...,N\} \tag{33}$$

where $\rho(S(k))$ denotes the spectral radius of $S(k)$.

Proof:

Existence of an NE:

We have the following fundamental game theory result from [11].

Theorem 3: The strategic noncooperative game $\mathcal{G} = \{\mathcal{Q}, \{\mathcal{X}_q\}_{q\in\mathcal{Q}}, \{\Phi_q\}_{q\in\mathcal{Q}}\}$ admits at least one NE if, for all $q \in \mathcal{Q}$: 1) the set $\mathcal{X}_q$ is an nonempty compact convex subset of a Euclidean space, and 2) the payoff function $\Phi_q(x)$ is continuous on $\mathcal{X}$ and quasi-concave on $\mathcal{X}_q$.

Since 1) the set $\mathcal{P}_q$ of the strategies, given by (16), is convex and compact and 2) the payoff function of each player, given by(15), is continuous in $\mathbf{p}$ and concave in $\mathbf{p}_q \in \mathcal{P}_q$ for any given $\mathbf{p}_{-q}$, game $\mathcal{G}$ always admits at least one NE.

Uniqueness of the NE:

This proof is based on the problem simplification and the result of K-matrix, thus the following intermediate results are need:

Definition 3: A matrix $A \in \mathcal{R}^{N\times N}$ is said to be a Z-matrix if its off-diagonal entries are all non-positive. A matrix $A \in \mathcal{R}^{N\times N}$ is said to be a P-matrix if all its principal minors are positive. A Z-matrix that is also P-matrix is called a K-matrix.

Lemma 2: A matrix $A \in \mathcal{R}^{N\times N}$ is a P-matrix if and only if A does not reserse the sign of any nonzero vector, i.e.

$$x_i[Ax]_i \leq 0, \text{ for all } i \Rightarrow x = 0. \tag{34}$$

Lemma 3: Let $A \in \mathcal{R}^{N\times N}$ be a K-matrix and B a nonnegative matrix. Then $\rho(A^{-1}B) < 1$ if and only if $A - B$ is a K-matrix.

The idea of the proof is that we first derive a necessary condition for two admissible strategies to be different NE of game $\mathcal{G}$. Then we obtain a sufficient condition that guarantees the previous condition is not satisfied; hence guaranteeing that there cannot be two different NE.

Assume game $\mathcal{G}$ admits two different NE points, denoted by $\mathbf{p}^{(0)}$ and $\mathbf{p}^{(1)}$, where $\mathbf{p}^{(r)} = [\mathbf{p}_1^{(r)T}, ..., \mathbf{p}_Q^{(r)T}]^T$ amd $\mathbf{p}_q^{(r)T} = [p_q^{(r)}(1), ..., p_q^{(r)}(N)]^T$, with $r = 0, 1$, and $q \in \mathcal{Q}$. From (16), the constraints of game $\mathcal{G}$ can be writen as $f_{q,k}(\mathbf{p}_q^{(r)}) > 0$, and

$$f_{q,k}(\mathbf{p}_q^{(r)}) = \begin{cases} p_q^{(r)}(k), & \text{if } 1 \leq k \leq N \\ 1 - \mathbf{1}^T\mathbf{p}_q^{(r)}, & \text{if } k = N+1 \\ p_q^{max}(k-N-1) - p_q^{(r)}(k-N-1), & \text{if otherwise} \end{cases} \tag{35}$$

From the objective function (15), we have

$$[\nabla_{\mathbf{p}_q} R_q(\mathbf{p})]_k \quad = \quad \frac{loge}{\Gamma_q N} \frac{|H_{qq}(k)|^2}{1 + \sum_{r \neq q} |H_{rq}(k)|^2 p_r(k) + \Gamma_q^{-1} |H_{qq}(k)|^2 p_q(k)} \tag{36}$$

So the Karush-Kuhn-Tucker (KKT) conditions needed to be satisfied are
1) primal constraints: $f_{q,k}(\mathbf{p}_q^{(r)}) \geq 0$
2) dual constraints: $\boldsymbol{\mu}_q^r \geq \mathbf{0}$
3) complementary slackness: $\mu_{q,k}^r f_{q,k}(\mathbf{p}_q^{(r)}) = 0$
4) gradient of Lagrangian with respect to $\mathbf{p}_q$ vanishes: $\nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(r)}) + \sum_{k=1}^{2N+1} \mu_{q,k}^{(r)} \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(r)}) = \mathbf{0}$
Then we have:

$$\begin{aligned}
& (\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T [\nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(0)}) + \sum_{k=1}^{2N+1} \mu_{q,k}^{(0)} \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(0)})] \\
& + (\mathbf{p}_q^{(0)} - \mathbf{p}_q^{(1)})^T [\nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(1)}) + \sum_{k=1}^{2N+1} \mu_{q,k}^{(1)} \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(1)})] \\
= \; & (\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T \nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(0)}) + (\mathbf{p}_q^{(0)} - \mathbf{p}_q^{(1)})^T \nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(1)}) \\
& + \sum_{k=1}^{2N+1} [\mu_{q,k}^{(0)} (\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(0)}) + \mu_{q,k}^{(1)} (\mathbf{p}_q^{(0)} - \mathbf{p}_q^{(1)})^T \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(1)})] = 0
\end{aligned} \tag{37}$$

Since $f_{q,k}(\mathbf{p}_q^{(r)})$ is linear, we have

$$\begin{aligned}
(\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(0)}) &= f_{q,k}(\mathbf{p}_q^{(1)}) - f_{q,k}(\mathbf{p}_q^{(0)}) \\
(\mathbf{p}_q^{(0)} - \mathbf{p}_q^{(1)})^T \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(1)}) &= f_{q,k}(\mathbf{p}_q^{(0)}) - f_{q,k}(\mathbf{p}_q^{(1)})
\end{aligned} \tag{38}$$

Thus the second term in (37) is zero and we get

$$\begin{aligned}
& (\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T \nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(0)}) + (\mathbf{p}_q^{(0)} - \mathbf{p}_q^{(1)})^T \nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(1)}) \\
= \; & -\sum_{k=1}^{2N+1} [\mu_{q,k}^{(0)} (\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(0)}) + \mu_{q,k}^{(1)} (\mathbf{p}_q^{(0)} - \mathbf{p}_q^{(1)})^T \nabla_{\mathbf{p}_q} f_{q,k}(\mathbf{p}_q^{(1)})] \\
= \; & -\sum_{k=1}^{2N+1} [\mu_{q,k}^{(0)} f_{q,k}(\mathbf{p}_q^{(1)}) + \mu_{q,k}^{(1)} f_{q,k}(\mathbf{p}_q^{(0)})] \leq 0
\end{aligned} \tag{39}$$

(39) is the necessary condition for two admissible strategies to be different NE. If (39) is strictly positive, then $\mathbf{p}^{(0)}$ and $\mathbf{p}^{(1)}$ cannot be different. So for some $q \in \mathcal{Q}$

$$\begin{aligned}
& (\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T \nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(0)}) + (\mathbf{p}_q^{(0)} - \mathbf{p}_q^{(1)})^T \nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(1)}) \\
= \; & (\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(0)})^T (\nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(0)}) - \nabla_{\mathbf{p}_q} R_q(\mathbf{p}^{(1)})) \\
= \; & \sum_{k \in \mathcal{D}_q} \{\alpha(k, \mathbf{p}^{(0)}, \mathbf{p}^{(1)}) |H_{qq}(k)|^2 (p_q^{(1)}(k) - p_q^{(0)}(k)) \\
& \times [\Gamma_q^{-1} |H_{qq}(k)|^2 + \sum_{r \neq q} |H_{rq}(k)|^2] (p_q^{(1)}(k) - p_q^{(0)}(k))\} > 0
\end{aligned} \tag{40}$$

where $\alpha(k, \mathbf{p}^{(0)}, \mathbf{p}^{(1)}) = \frac{loge}{\Gamma_q N} (1 + \sum_{r \neq q} |H_{rq}(k)|^2 p_r^{(0)}(k) + \Gamma_q^{-1} |H_{qq}(k)|^2 p_q^{(0)}(k))^{-1} (1 + \sum_{r \neq q} |H_{rq}(k)|^2 p_r^{(1)}(k) + \Gamma_q^{-1} |H_{qq}(k)|^2 p_q^{(1)}(k))^{-1} > 0$. Define $\mathcal{K}_q$ as the set of carriers in $\mathcal{D}_q$ such that the two solutions coincide and observe that $\mathcal{K}_q \neq \mathcal{D}_q$. From (40), we have

$$(p_q^{(1)}(k) - p_q^{(0)}(k)) [\Gamma_q^{-1} |H_{qq}(k)|^2 + \sum_{r \neq q} |H_{rq}(k)|^2] (p_q^{(1)}(k) - p_q^{(0)}(k))$$

$$
\begin{aligned}
&= \Gamma_q^{-1}|H_{qq}(k)|^2(p_q^{(1)}(k)-p_q^{(0)}(k))^2 + (p_q^{(1)}(k)-p_q^{(0)}(k))\sum_{r\neq q}|H_{rq}(k)|^2(p_q^{(1)}(k)-p_q^{(0)}(k)) \\
&= \Gamma_q^{-1}|H_{qq}(k)|^2|p_q^{(1)}(k)-p_q^{(0)}(k)| + sign(p_q^{(1)}(k)-p_q^{(0)}(k))\sum_{r\neq q}|H_{rq}(k)|^2(p_q^{(1)}(k)-p_q^{(0)}(k)) \\
&> 0, \forall k \in \mathcal{D}_q\backslash\mathcal{K}_q \text{ and some } q \in \mathcal{Q} \qquad (41)
\end{aligned}
$$

where $sign(.)$ is the sign function, defined as $sign(x) = 1$ if $x > 0$, $sign(x) = 0$ if $x = 0$, and $sign(x) = -1$ if $x < 0$. Since $(p_r^{(1)}(k) - p_r^{(0)}(k)) = 0$ whenever $k \notin \mathcal{D}_r$, so

$$
\begin{aligned}
|p_q^{(1)}(k)-p_q^{(0)}(k)| + sign(p_q^{(1)}(k)-p_q^{(0)}(k))\sum_{r\neq q}G_{rq}(k)(p_q^{(1)}(k)-p_q^{(0)}(k)) \quad &> \quad 0 \\
\forall k \in \mathcal{D}_q\backslash\mathcal{K}_q \text{ and some } q \in \mathcal{Q} \qquad (42)
\end{aligned}
$$

where

$$
G_{rq}(k) = \begin{cases} \Gamma_q \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2}, & \text{if } k \in \mathcal{D}_r, \\ 0, & \text{otherwise.} \end{cases} \qquad (43)
$$

Define $\Delta_q(k) = |p_q^{(1)}(k) - p_q^{(0)}(k)|$ and considering the worst possible case, we get

$$
\Delta_q(k) > \sum_{r\neq q} G_{rq}(k)\Delta_r(k), \forall k \in \mathcal{D}_q\backslash\mathcal{K}_q \text{ and some } q \in \mathcal{Q} \qquad (44)
$$

when (44) is satisfied, $\mathbf{p}^{(0)}$ and $\mathbf{p}^{(1)}$ cannot be different. If the problem has two different NE, the following condition needs to be satistied by such $\mathbf{p}^{(0)}$ and $\mathbf{p}^{(1)}$:

$$
\Delta_q(k) \leq \sum_{r\neq q} G_{rq}(k)\Delta_r(k), \forall k \in \mathcal{D}_q\backslash\mathcal{K}_q \text{ and some } q \in \mathcal{Q} \qquad (45)
$$

Since $\Delta_q(k) = 0$ whenever $k \notin \mathcal{D}_q$ and from (43), we introduce

$$
\tilde{G}_{rq}(k) = \begin{cases} \Gamma_q \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2}, & \text{if } k \in \mathcal{D}_q \cap \mathcal{D}_r, \\ 0, & \text{otherwise.} \end{cases} \qquad (46)
$$

and we have

$$
\Delta_q(k_q) \leq \sum_{r\neq q} \tilde{G}_{rq}(k_q)\Delta_r(k_q), \forall k_q \in \mathcal{D}_q\backslash\mathcal{K}_q \text{ and some } q \in \mathcal{Q} \qquad (47)
$$

where k_q denotes any subcarrier index in the set $\mathcal{D}_q\backslash\mathcal{K}_q$. Condition (47) can be rewritten in a vector form

$$
(I - \bar{S})\Delta \leq 0 \qquad (48)
$$

where

$$
\begin{aligned}
\Delta &= [\Delta^T(1), ..., \Delta^T(NQ)]^T && (49) \\
\Delta(k) &= [\Delta_1(k), ..., \Delta_Q(k)]^T && (50) \\
\bar{S} &= diag(\{\bar{S}(k)\}_{k=1}^N) && (51)
\end{aligned}
$$

$$
[\bar{S}(k)]_{qr} = \begin{cases} \tilde{G}_{rq}(k), & \text{if } k = k_q \text{ and } r \neq q, \\ 0, & \text{otherwise.} \end{cases} \qquad (52)
$$

Since $\Delta \geq 0$, (48) implies $\Delta_i[(I - \bar{S})\Delta]_i \leq 0$. From *Lemma 2*, we know

$$
I - \bar{S} \text{ is a P-matrix} \qquad (53)
$$

From (32), we see

$$
[S(k)]_{qr} = \begin{cases} \tilde{G}_{rq}(k), & \text{if } r \neq q, \\ 0, & \text{otherwise.} \end{cases} \qquad (54)
$$

From *Definition 3*, we know $I - \bar{S}$ and $I - S$ are both Z-matrices and $I - \bar{S} \geq I - S$ componentwisely. So a sufficient condition for (53) is [12]:

$$I - S \text{ is a P-matrix} \tag{55}$$

So $I - S$ is a K-matrix from *Definition 3*. By *Lemma 3*, we have

$$\rho(S) < 1 \tag{56}$$

which leads to (33). This completes the proof.

B Proof by LCP Formulation [7]

B.1 Background on LCP and AVI

This proof requires the background on Linear Complementarity Problem (LCP) and Affine Variational Inequalities (AVI). The following definitions and results are needed [12, 13].

Definition 4 [12]: Given a vector $q \in \mathcal{R}^n$ and a matrix $M \in \mathcal{R}^{n\times n}$, the *linear complementarity problem*, abbreviated LCP, is to find a vector $x \in \mathcal{R}^n$ such that

$$x \geq 0 \tag{57}$$

$$q + Mx \geq 0 \tag{58}$$

$$x^T(q + Mx) = 0 \tag{59}$$

or to show that no such vector x exists.

Definition 5 [13]: Given a subset $\mathcal{K}$ of the Euclidean n-dimensional space $\mathcal{R}^n$ and a mapping $F : \mathcal{K} \to \mathcal{R}^n$, the *variational inequality*, denoted $VI(\mathcal{K}, F)$, is to find a vector $x \in \mathcal{K}$ such that

$$(y - x)^T F(x) \geq 0, \forall y \in \mathcal{K} \tag{60}$$

The set of solutions to this problem is denoted $SOL(\mathcal{K}, F)$. When F is the affine function given by:

$$F(x) = q + Mx, \forall x \in \mathcal{R}^n \tag{61}$$

for some vector $q \in \mathcal{R}^n$ and matrix $M \in \mathcal{R}^{n\times n}$ and $\mathcal{K}$ is a polyhedral set, the *affine variational inequality*, denoted $AVI(\mathcal{K}, q, M)$, is to find a vector $x \in \mathcal{K}$ such that

$$(y - x)^T F(x) = (y - x)^T (q + Mx) \geq 0, \forall y \in \mathcal{K} \tag{62}$$

In a general noncooperative game defined in (66), the following result gives a set of sufficient conditions under which a NE can be obtained by solving a VI.

Proposition 1 [13]: Let each $\mathcal{K}_i$ be a closed convex subset of $\mathcal{R}^{n_i}$. Suppose that for each fixed tuple $\tilde{\mathbf{x}}^i$, the function $\theta_i(x^i, \tilde{\mathbf{x}}^i)$ is convex and continuously differentiable in x^i. Then a tuple $\mathbf{x} = (x^i : i = 1, ..., N)$ is a NE if and only if $\mathbf{x} \in SOL(\mathcal{K}, F)$, where

$$\mathcal{K} = \prod_{i=1}^{N} \mathcal{K}_i \text{ and } F(\mathbf{x}) = (\nabla_{x_i}\theta_i(\mathbf{x}))_{i=1}^N \tag{63}$$

i.e. $\mathbf{x}$ is a NE if and only if for each player $i = 1, ..., N$,

$$(y^i - x^i)^T \nabla_{x_i}\theta_i(\mathbf{x}) \geq 0, \forall y^i \in \mathcal{K}_i \tag{64}$$

Definition 6 [13]: If $\mathcal{K} = \prod_i^N \mathcal{K}_i$, a map $F : \mathcal{K} \to \mathcal{R}^n$ is a *uniformly P function* on $\mathcal{K}$ if there exists a constant $\mu > 0$ such that for all pairs of vectors x and y in $\mathcal{K}$,

$$max_{1\leq i\leq N}(x_i - y_i)T(F_i(x) - F_i(y)) \geq \mu\|x - y\|_2^2 \tag{65}$$

Proposition 2 [13]: If $\mathcal{K} = \prod_i^N \mathcal{K}_i$ and each $\mathcal{K}_i$ is closed convex and F is a continuous uniformly P function on $\mathcal{K}$, then the $VI(\mathcal{K}, F)$ has a unique solution.

B.2 Problem Reformulation

The specific problem formulation is: for $k = 1, ..., n$

$$\begin{aligned} \text{maximize}_{\{p^i\}} \quad & f_i(p^1, ..., p^m) = \sum_{k=1}^{n} log(1 + \frac{p_k^i}{\sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} p_k^j}) \\ \text{subject to} \quad & p^i \in \mathcal{P}^i \end{aligned} \tag{66}$$

where σ_k^i are positive scalars representing noise power spectral; α_k^{ij} are nonnegative scalars for all $i \neq j$ representing channel crosstalk coefficients; and

$$\mathcal{P}^i = \{p^i \in \mathcal{R}^n | 0 \leq p_k^i \leq CAP_k^i, \forall k = 1, ..., n, \sum_{k=1}^{n} p_k^i \leq P_{max}^i\} \tag{67}$$

In [7], it is assumed that $\alpha_k^{ii} = 1$ for all i and k. To avoid triviality, it is assumed that

$$P_{max}^i < \sum_{k=1}^{n} CAP_k^i \tag{68}$$

which ensures that the power constraint $\sum_{k=1}^{n} p_k^i \leq P_{max}^i$ is not redundant.

The KKT conditions for this problem are: $\forall k = 1, ..., n$

1) primal constraints:

$$P_{max}^i - \sum_{k=1}^{n} p_k^i \geq 0 \tag{69}$$

$$CAP_k^i - p_k^i \geq 0 \tag{70}$$

$$p_k^i \geq 0 \tag{71}$$

2) dual constraints:

$$u_i \geq 0 \tag{72}$$

$$\gamma_k^i \geq 0 \tag{73}$$

$$\lambda_k^i \geq 0 \tag{74}$$

3) complementary slackness:

$$u_i(P_{max}^i - \sum_{k=1}^{n} p_k^i) = 0 \tag{75}$$

$$\gamma_k^i(CAP_k^i - p_k^i) = 0 \tag{76}$$

$$\lambda_k^i(p_k^i) = 0 \tag{77}$$

4) gradient of Lagrangian with respect to p_i:

$$\nabla_{p_i}[f_i(p^1, ..., p^m) + u_i(P_{max}^i - \sum_{k=1}^{n} p_k^i) + \gamma_k^i(CAP_k^i - p_k^i) + \lambda_k^i(p_k^i)]$$

$$= \frac{1}{\sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} p_k^j + p_k^i} - u_i - \gamma_k^i + \lambda_k^i \tag{78}$$

$$= \frac{1}{\sigma_k^i + \sum_{j=1}^{m} \alpha_k^{ij} p_k^j} - u_i - \gamma_k^i + \lambda_k^i = 0 \tag{79}$$

From (74)(77)(79), we have

$$-\frac{1}{\sigma_k^i + \sum_{j=1}^{m} \alpha_k^{ij} p_k^j} + u_i + \gamma_k^i \geq 0 \text{ and } (-\frac{1}{\sigma_k^i + \sum_{j=1}^{m} \alpha_k^{ij} p_k^j} + u_i + \gamma_k^i)p_k^i = 0 \tag{80}$$

Though the KKT system is nonlinear, it can be shown that under the assumption (68), the system is equivalent to a mixed LCP system.

Proposition 3 [7]: Suppose that (68) holds. The above KKT condition is equivalent to: $\forall k = 1, ..., n$

$$\begin{aligned} & 0 \le p_k^i \perp \sigma_k^i + \sum_{j=1}^{m} \alpha_k^{ij} p_k^j + v_i + \varphi_k^i \ge 0 \\ & v_i \text{ free}, \ P_{max}^i - \sum_{k=1}^{n} p_k^i = 0 \\ & 0 \le \varphi_k^i \perp CAP_k^i - p_k^i \ge 0 \end{aligned} \tag{81}$$

Proof:

If $u_i = 0$, then $\gamma_k^i \ge (\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j)^{-1} > 0$, which implies $CAP_k^i - p_k^i = 0$, thus $P_{max}^i \ge \sum_{k=1}^n p_k^i = \sum_{k=1}^n CAP_k^i$. This contradicts (68). So $u_i \ne 0$ and $P_{max}^i - \sum_{k=1}^n p_k^i = 0$. Define

$$v_i = -\frac{1}{u_i} \text{ (so } v_i \text{ is free)} \tag{82}$$

$$\varphi_k^i = \frac{\gamma_k^i (\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j)}{u_i} \tag{83}$$

Then substituting (82) and (83) back to (81), we can easily see that the KKT system holds. This completes the proof.

The mixed LCP (81) is actually the KKT condition of the $AVI(\mathcal{X}, \sigma, M)$ defined by the affine mapping $p = (p^i)_{i=1}^m \in \mathcal{R}^{mn} \to \sigma + Mp \in \mathcal{R}^{mn}$ and the polyhedron $\mathcal{X} = \prod_{i=1}^m \widehat{\mathcal{P}}^i$, where $\sigma = (\sigma_k^i)_{i=1}^m$, M is the block partitioned matrix $(M^{ij})_{i,j=1}^m$ with each $M^{ij} = diag(\alpha_k^{ij})_{k=1}^n$ and

$$\widehat{\mathcal{P}}^i = \{p^i \in \mathcal{R}^n | 0 \le p_k^i \le CAP_k^i, \forall k = 1, ..., n, \sum_{k=1}^n p_k^i = P_{max}^i\} \tag{84}$$

According to *Proposition 1*, the tuple p is a NE if and only if $p \in \mathcal{X}$ and

$$(\tilde{p} - p)^T (\sigma + Mp) \ge 0, \forall \tilde{p} \in \mathcal{X} \tag{85}$$

B.3 Uniqueness Conditions and the Proof

Define the $m \times m$ matrix $B = [b_{ij}]$ by

$$b_{ij} = max_{1 \le k \le n} \alpha_k^{ij}, \forall i, j = 1, ..., m \tag{86}$$

Note that $b_{ii} = 1$. Let B_{dia}, B_{low} and B^{upp} be the diagonal, strictly lower, and strictly upper triangular parts of B, respectively. Since α_k^{ij} are all nonnegative, B is a nonnegative matrix. From *Definition 3* and all principal minors of $B_{dia} - B_{low}$ are equal to one, we know that $B_{dia} - B_{low}$ is a M-matrix and thus $(B_{dia} - B_{low})^{-1}$ exists and is a nonnegative matrix. Therefore, so is the matrix

$$\Upsilon = (B_{dia} - B_{low})^{-1} B^{upp}. \tag{87}$$

The matrix

$$\bar{B} = B_{dia} - B_{low} - B^{upp} \tag{88}$$

is the "comparison matrix" of B. Note that $\bar{B}$ is a Z-matrix. B is called a H-matrix if $\bar{B}$ is also a P-matrix. Two characterizations hold for the latter condition: a) $\rho(\Upsilon) < 1$ and b) for every nonzero vector $x \in \mathcal{R}^m$, there exists an index i such that $x_i(\bar{B}x)_i > 0$.

Proposition 4 [7]: Suppose that

$$max_{1 \le i \le m} \sum_{k=1}^n \sum_{j=1}^m \alpha_k^{ij} p_k^i p_k^j > 0, \forall p = (p^i)_{i=1}^m \ne 0 \tag{89}$$

There exists a unique NE. In particular, this holds if either one of the following two conditions is satisfied: (a) for every $k = 1, ..., n$, the tone matrix M_k is positive definite, where $(M_k)_{ij} = \alpha_k^{ij}, \forall i, j = 1, ..., m$; (b) $\rho(\Upsilon) < 1$.

Proof:

From *Proposition 2*, $AVI(\mathcal{X}, \sigma, M)$ has a unique solution if M has the uniform P property, thus for any nonzero tuple $p = (p^i)_{i=q}^m$, $max_{1 \leq i \leq m} (p^i)^T \sum_{j=1}^m M^{ij} p^j > 0$, which is precisely (89).

For condition (a), M is positive definite because it is a principal rearrangement of $diag(M_k)_{k=1}^n$. So $p^T M p = \sum_{i=1}^m \sum_{k=1}^n \sum_{j=1}^m \alpha_k^{ij} p_k^i p_k^j > 0$.

For condition (b),

$$
\begin{aligned}
\sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_k^{ij} p_k^i p_k^j &= \sum_{k=1}^{n} (p_k^i)^2 + \sum_{j \neq i} \sum_{k=1}^{n} \alpha_k^{ij} p_k^i p_k^j \\
&\geq \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \neq i} \sum_{k=1}^{n} \alpha_k^{ij} |p_k^i| |p_k^j| \\
&\geq \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \neq i} (\sum_{k=1}^{n} (p_k^i)^2)^{1/2} (\sum_{k=1}^{n} (\alpha_k^{ij} p_k^j)^2)^{1/2} \text{ (Cauchy-Schwarz inequality)} \\
&\geq \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \neq i} max_{1 \leq k \leq n} \alpha_k^{ij} (\sum_{k=1}^{n} (p_k^i)^2)^{1/2} (\sum_{k=1}^{n} (p_k^j)^2)^{1/2} \\
&= (\sum_{k=1}^{n} (p_k^i)^2)^{1/2} \sum_{j=1}^{m} \bar{b}_{ij} (\sum_{k=1}^{n} (p_k^j)^2)^{1/2} \text{(from (88))} \qquad (90)
\end{aligned}
$$

Defining $q_i = (\sum_{k=1}^n (p_k^i)^2)^{1/2}$, we have $\sum_{j=1}^m \sum_{k=1}^n \alpha_k^{ij} p_k^i p_k^j \geq q_i \sum_{j=1}^m \bar{b}_{ij} q_j = q_i (\bar{B} q)_i, \forall i = 1, ..., m$. Since condition (b) $\rho(\Upsilon) < 1$ implies that $max_{1 \leq i \leq m} q_i (\bar{B} q)_i > 0$, thus (89) holds. This completes the proof.

V Comparison of the Results

The noncooperative game always admits a NE and the result is based on the game theory [11], so no comparison on this is needed.

The sufficient conditions for the uniqueness of the NE derived from [7] and [8] are based on two different methods. The proof in [8] is long and I find that it is easy to make mistakes when I read the proof. The result, however, is very good, and gives additional insight into the physical interpretation of the conditions for the uniqueness of the NE, i.e. the uniqueness of NE is ensured if the links are sufficiently far apart from each other. [8] actually give two detailed physical interpretation of NE on low-interference and high interference case, respectively. It also makes some investigation on the efficiency of the NE. The proof in [7] is short and neat, as it takes the advantages of reformulating the game by LCP and AVI, thus many results for LCP and AVI can be applied. However, the result does not give a clear insight into the physical interpretation. Furthermore, LCP and AVI are real-valued, as mentioned in *Definition 4 & 5*. The noise covariance (σ) and the channel model (M) may be complex-valued, thus the result may not be applicable.

Both [7] and [8] generalize and extend the results in other papers [1, 2, 3, 4, 5, 6], which have fewer constraints, such as spectral mask constraint, and more restrictive conditions. [8] even give specific corresponding conditions using the new result for previous works. [1, 2, 3] actually talk about the same thing, just on different channels, i.e. Gaussian interference channel, VDSL and DSL, respectively. They consider the two-user system, which is generalized to the system with arbitrary number of users by other papers. [4, 5, 6] consider the system having arbitrary number of users. But they do not include spectral mask constraint, which is practical.

[8] discusses two optimization problems and proves their equivalence, which is unique among those papers. It proves that diagonal transmission from each user through the channel eigenmodes is optimal,

irrespective of the channel state, power budget, spectral mask constraints, and interference levels, which yields a strong simplification of the original optimization.

VI Conclusion

This report considers the application of convex optimization on design of multiuser non-cooperative communication systems via game theory and focuses on the problems on existence and uniqueness of the NE solution. All the eight papers are based on single-input single-output systems. I do not have enough time to investigate the result for multiple-input multiple-output systems, such as [14]. A further question after the study on existence and uniqueness of the NE is that: how to find the NE point? Iterative and distributed algorithms are applied. [7] and [8] have shown that the uniqueness condition is actually also responsible for the convergence of the algorithms. As this project is part of my Final Year Thesis project, uniqueness condition for MIMO system, detailed algorithms and their convergence will be studied afterwards. Further extension lies in the dealing with imperfect channel state information or other new algorithms. I will see whether I can propose a algorithm and prove its convergence.

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