Final Year Thesis

Design of Multiuser Non-Cooperative Communication Systems via Game Theory

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Abstract

This project considers the maximization of information rate on each link in the multiuser non-cooperative communication system given the constraints on the transmit power. By using a game theoretical approach, the interference channel can be modeled as a non-cooperative power competing game among different users. The Nash Equilibrium (NE) of the game gives the solution. There are three questions to ask: 1) whether the NE exists? 2) if it exists, whether it is unique? 3) if it is unique, how to find the equilibrium point? This problem has been studied in a number of papers. A variety of conditions guaranteeing the existence and uniqueness of the NE has been given. The convergence of many different distributed algorithms have also been derived. In this project, I proposed Partially Asynchronous Iterative Waterfilling Algorithm to solve the problem. The mathematical proof of its convergence and the numerical simulations comparing with existing approaches have also been derived.

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Contents

List of Figures

Chapter 1

Introduction

The interference channel is a mathematical model relevant to many physical communication channels and multiuser systems where multiple uncoordinated links share a common communication medium. It is characterized by its capacity region, defined as the set of rates that can be simultaneously achieved by the users in the system. In principle, this multi-objective optimization of the communication system requires a centralized solution, which has high complexity, heavy signaling and the coordination among the users. Instead, the system can be designed in a game theoretical approach in a fully distributed fashion with no centralized control. The basic idea is that users in the system can be modeled as several players having different objectives, say maximizing their own information rate, and behave selfishly without cooperation. Therefore, the original multi-objective optimization problem is converted into a set of mutually coupled single-objective optimization problems. The optimal solution to this approach is the achievement of Nash Equilibrium (NE)[1]. Then iterative algorithms[2] can be utilized to solve this kind of problems. Several researches have been done in this area. This project proposed an alternative iterative distributed algorithm to solve the problem. The condition guaranteeing the convergence of the algorithm has been derived.

1.1 Related Work

From the communication system view, several communication systems with different requirements and constraints have been studied. For Single-Input Single-Output (SISO) systems, current work in the field can be divided in three large classes, according to the kind of games dealt with: scalar, vector and matrix-valued power control games. Users in scalar games have one degree of freedom to optimize their transmit power or rate. Based on the standard function proposed in [3], solutions have been provided in [4, 5, 6, 7, 8]. This kind of problem can also be recast as convex optimization problems [9]. For vector games, it is more complicated as each user has several degrees of freedom to maximize, e.g. power allocation across frequency bins. Since the seminal paper [10] studying the maximization of the information rate of two-user SISO Digital Subscriber Lines (DSL) system by strategic non-cooperative power competing game modeling, a number of researches have been done for the case of SISO frequency-selective channels

[11, 12, 13, 14]. In [15, 16], the problem was extended to matrix-valued games together with spectral mask constraints. It is proved that complicated matrix-valued problems can be converted into unified vector power control game with no performance penalty. For more general Multiple-Input Multiple-Output (MIMO) cases, a two-user Multiple-Input Single-Output (MISO) channel was studied [17]. Rate maximization game in MIMO interference channels was studied in [18, 19, 20, 21]. Mutual information maximization in MIMO Gaussian interference channel was studied in [22] together with the conditions that guarantee the uniqueness of the NE of the MIMO game and the convergence of the proposed distributed algorithms. However, those results above for MIMO systems are valid only for nonsingular square channel matrices. [23] provides a complete characterization of the MIMO game for arbitrary channel matrices. Conditions guaranteeing both the uniqueness of the NE and the convergence of the algorithms have been derived.

From distributed algorithm view, three major schemes of the distributed algorithm have been considered according to the kinds of updating schedule [2]: Gauss-Seidel scheme (i.e. synchronous sequential) [10, 11, 12, 13, 14], Jacobi Scheme (i.e. synchronous simultaneously) [16, 24, 25] and totally asynchronous scheme [23, 26, 27]. Two iterative and distributed algorithms have been applied based on those three kinds of updating schemes, namely water-filling based algorithms [10, 11, 12, 13, 14, 16, 23, 24, 25, 26, 27] and gradient-projection based algorithms [16].

From the view of approaches used to analyze the power competing game, three key results are applied: 1) the interpretation of the waterfilling mapping as a projector [16, 22, 23]; 2) the interpretation of the NE of the game as the solution of a proper affine Variational Inequality (VI) problem [14]; and 3) the interpretation of the waterfilling mapping as a piecewise affine function [24].

1.2 Aim and Objectives

This project considers the maximization of information rate on each link in the multiuser non-cooperative communication system given the constraints on the transmit power. In order to avoid high complexity, heavy signaling and the coordination among the users required by the centralized solution, distributed algorithms are used from a game theoretical approach. The system model considered in this project is mainly based on the one discussed in [22]. MIMO interference channels with nonsingular square channel matrices are assumed. By using the result of the interpretation of the waterfilling mapping as a projector, I proposed the Partially Asynchronous Iterative Waterfilling Algorithm (IWFA), which is a variation of the totally asynchronous IWFA and is the general case for Gauss-Seidel, Jacobi and totally asynchronous schemes. Mathematical proof of the convergence as well as numerical simulations and comparisons were also derived.

1.3 Report Outline

The report is organized as follows. Chapter 2 gives the system model and problem formulation by game theoretical approach. Chapter 3 talks about the mathematical tools

I used for modeling and solving the problem, including game theory, convex optimization theory, fixed point theory, distributed algorithm theory and norm representation. In Chapter 4, contraction property of the waterfilling operator, which is crucial in proving the convergence of the distributed algorithm, is discussed. In Chapter 5, the conditions guaranteeing the existence and uniqueness of the NE for the power competing game are proved. Chapter 6 discusses the Partially Asynchronous IWFA and its convergence conditions in details. Chapter 7 reports some numerical results and comparisons with existing approaches. Finally, Chapter 8 draws some conclusions.

Chapter 2

System Model and Problem Formulation

In this chapter, a general system model is provided and the problem is formulated by game theoretical approach.

2.1 System Model

The general system model [22] considers a Gaussian interference channel with Q links. Each link is communicating through a MIMO channel with n_{T_q} transmitter and n_{R_q} receivers. It can be represented as follows:

$$y_q = H_{qq}x_q + \sum_{r \neq q} H_{rq}x_q + n_q, \tag{2.1}$$

where $x_q \in \mathcal{C}^{n_{T_q}}$ is the vector transmitted by source q; $y_q \in \mathcal{C}^{n_{R_q}}$ is the vector received by destination q; $H_{qq} \in \mathcal{C}^{n_{R_q} \times n_{T_q}}$ is the direct channel of link q and is assumed to be square nonsingular; $H_{rq} \in \mathcal{C}^{n_{R_q} \times n_{T_r}}$ is the cross-channel matrix between source r and destination q; and $n_q \in \mathcal{C}^{n_{R_q}}$ is the noise vector with covariance matrix R_{n_q}. $\sum_{r \neq q} H_{rq}x_q$ represents the multi-user interference (MUI) received by the qth destination and caused by other links; and it can be treated as additive noise. The total average transmit power for each link q is

$$\mathcal{E}\{\|x_q\|_2^2\} = Tr(Q_q) \leq P_q, \tag{2.2}$$

where $Q_q = \mathcal{E}\{x_q x_q^H\}$ is the covariance matrix of the transmitted power in units of energy per transmission and P_q is the power constraint for each link q.

This system model is sufficiently general to represent many cases of practical interest, such as digital subsciber line, cellular radio and ad hoc wireless networks. It assumes that each destination has perfect knowledge of the channel from its source, but not of the channel from the interfering sources.

The maximum information rate on link q for a given set of users' covariance matrices $Q_1, ..., Q_Q$ is [28]

$$R_q(Q_q, Q_{-q}) = \log\det(I + H_{qq}^H R_{-q}^{-1}(Q_{-q})H_{qq}Q_q) \tag{2.3}$$

where $R_{-q}(Q_{-q}) = R_{n_q} + \sum_{r \neq q} H_{rq} Q_r H_{rq}^H$ is the MUI plus noise covariance matrix observed by user q, and $Q_{-q} = (Q_r)_{r=1, r \neq q}^{Q}$ is the set of all the users' covariance matrices, except the q-th one.

2.2 Problem Formulation

The system can be formulated in a game theoretical approach [1]. In a general noncooperative game, there are N players, each of whom has a certain payoff function and strategy set that may depend on the other players' actions. Assume that player i's strategy set is $\mathcal{K}_i$, which is a subset of $\mathcal{R}^{n_i}$ and is independent of the other players' actions. Player i's payoff function $\theta_i(\mathbf{x})$ depends on all players' strategies, which are described by the vector $\mathbf{x}$ that consists of the subvectors $x^i \in \mathcal{R}^{n_i}$ for $i = 1, ..., N$. Player i's problem is to determine, for each fixed but arbitrary tuple $\tilde{\mathbf{x}}^i = (x^j : j \neq i)$ of other players' strategies, an optimal strategy x^i that solves the payoff maximization problem in the variable x^i:

$$\begin{array}{ll} \text{maximize}_{\{x^i\}} & \theta_i(x^i, \tilde{\mathbf{x}}^i) \\ \text{subject to} & x^i \in \mathcal{K}_i \end{array} \tag{2.4}$$

The solution set of the optimization problem is denoted by $S_i(\tilde{\mathbf{x}}^i)$. A Nash Equilibrium (NE) is a tuple of strategies $\mathbf{x} = (x^i : i = 1, ..., N)$ with the property that for each i, $x^i \in S_i(\tilde{\mathbf{x}}^i)$. In words, a NE is a tuple of strategies, one for each player, such that no player can enlarge the payoff by unilaterally deviating his action from his designated strategy.

In this system, the players are the Q links and the payoff functions are the information rates $R_q(Q_q, Q_{-q})$ on each link, where transmit covariance matrix Q_q is the strategy that can be used by user q. Thus, this system has the following structure:

$$(\mathcal{G}) \qquad \begin{array}{ll} \text{maximize}_{\{Q_q\}} & R_q(Q_q, Q_{-q}) \\ \text{subject to} & Q_q \in \mathcal{Q}_q, \quad \forall q \in \Omega \end{array} \tag{2.5}$$

where $\Omega = \{1, ..., Q\}$ is the set of players and $\mathcal{Q}_q$ is the set of admissible strategies for each player q:

$$\mathcal{Q}_q = \{Q \in \mathcal{C}^{n_{T_q} \times n_{T_q}} : Q \succeq 0, Tr\{Q\} = P_q\} \tag{2.6}$$

The solution of the game $\mathcal{G}$ is the NE.

Notice that $Q = Q^H$ since any complex positive semidefinite matrix must be necessarily hermitian[29]. Here $Tr\{Q\} = P_q$ was used instead of $Tr\{Q\} \leq P_q$ as originally mentioned in (2.2) because at the optimal solution of the game $\mathcal{G}$, the constraint must be satisfied with equality.

Definition 2.1. [1] A NE of a pure strategic game $\mathcal{G}$ is a profile $Q^* = (Q_q^*)_{q \in \Omega} \in \mathcal{Q}_1 \times ... \times \mathcal{Q}_Q$ with the property that

$$R_q(Q_q^*, Q_{-q}^*) \geq R_q(Q_q, Q_{-q}^*), \qquad \forall Q_q \in \mathcal{Q}_q, \forall q \in \Omega \tag{2.7}$$

□

Definition 2.2. [1] A NE of a mixed strategic game $\bar{\mathcal{G}}$ is a profile $\bar{Q}^* = (\bar{Q}_q^*)_{q\in\Omega} \in \bar{\mathcal{Q}}_1 \times ... \times \bar{\mathcal{Q}}_Q$ with the property that

$$E_{f_{Q_q}} E_{f_{Q_{-q}}} \{R_q(Q_q^*, Q_{-q}^*)\} \geq E_{f_{Q_q}} E_{f_{Q_{-q}}} \{R_q(Q_q, Q_{-q}^*)\}, \qquad \forall Q_q \in \mathcal{Q}_q, \forall q \in \Omega \tag{2.8}$$

where $\bar{\mathcal{G}} = \{\Omega, \{\bar{\mathcal{Q}}_q\}_{q\in\Omega}, \{\bar{R}_q\}_{q\in\Omega}\}$; $\bar{\mathcal{Q}}_q$ denotes the set of the probability distributions over the set $\mathcal{Q}_q$ of pure strategies; $E_{f_{Q_q}} E_{f_{Q_{-q}}}(R_q)$ is the expectation of R_q over the mixed strategies of all players; and $f_{Q_q}(Q_q)$ is the probability density function.

□

We can actually limit ourselves to adopt pure strategies only. Given Jensen's inequality [28] and the concavity of the function $R_q(Q_q, Q_{-q})$, we have

$$E_{f_{Q_q}} E_{f_{Q_{-q}}} \{R_q(Q_q, Q_{-q})\} \leq E_{f_{Q_{-q}}} \{R_q(E_{f_{Q_q}}\{Q_q\}, Q_{-q})\}, \forall Q_q \in \mathcal{Q}_q, \forall q \in \Omega \tag{2.9}$$

The equality holds if and only if Q_q reduces to a pure strategy. Therefore, whatever the strategies of the other players are, every NE of the game is achieved using pure strategies [15].

Given $q \in \Omega$ and $Q_{-q} \in \mathcal{Q}_{-q} = \mathcal{Q}_1 \times ... \times \mathcal{Q}_{q-1}, \mathcal{Q}_{q+1} \times ... \times \mathcal{Q}_Q$, the solution to problem (2.5) is the waterfilling solution [28]:

$$Q_q^* = WF_q(Q_{-q}) \tag{2.10}$$

By the eigendecomposition of $H_{qq}^H R_{-q}^{-1}(Q_{-q}) H_{qq} = U_q D_q U_q^H$ from equation (2.3), where $U_q = U_q(Q_{-q}) \in \mathcal{C}^{n_{T_q} \times n_{T_q}}$ is a unitary matrix with the eigenvectors and $D_q = D_q(Q_{-q}) \in \mathcal{R}_{++}^{n_{T_q} \times n_{T_q}}$ is a diagonal matrix with n_{T_q} positive eigenvalues, the waterfilling operator $WF(.)$ can be written as

$$WF_q(Q_{-q}) = U_q(\mu_q I - D_q^{-1})^+ U_q^H \tag{2.11}$$

where μ_q is the waterlevel satisfying $Tr\{(\mu_q I - D_q^{-1})^+\} = P_q$ with $(x)^+ = \max(0, x)$.

By Definition 2.1 and waterfilling solution (2.10), the NE of the game $\mathcal{G}$ can be charachterized as the waterfilling fixed-point equation:

$$Q_q^* = WF_q(Q_{-q}^*), \quad \forall q \in \Omega \tag{2.12}$$

To study game $\mathcal{G}$, the fixed-point mapping of the NE and an alternative interpretation of the MIMO waterfilling solution as a proper projector operator are crucial for the derivation of the conditions for the existence and uniqueness of the NE and the convergence of the partially asynchronous distributed algorithm. In Chapter 3, mathematical tools that we are going to use to analyze the game will be discussed.

Chapter 3

Mathematical Tools

This chapter talks about the mathematical tools used to analyze the strategic power competing game. The standard results from game theory, convex optimization theory, fixed-point theory, distributed algorithm theory and Norm Representation are very important for the derivation of the conditions for the existence and uniqueness of the NE and the convergence of the partially asynchronous distributed algorithm.

3.1 Game Theory

Theorem 3.1. *[1, 30] The strategic noncooperative game $\mathcal{G} = \{\Omega, \{\mathcal{Q}_q\}_{q\in\Omega}, \{R_q\}_{q\in\Omega}\}$ admits at least one NE if, for all $q \in \Omega$: 1) the set $\mathcal{Q}_q$ of feasible strategy profiles is compact and convex; and 2) the payoff function R_q is continuous on $Q \in \mathcal{Q}$ and concave in $Q_q \in \mathcal{Q}_q$, for any given $Q_{-q} \in \mathcal{Q}_{-q}$. (This theorem is formulated by the notations used in this report.)*

□

Theorem 3.1 will be used to prove the existence of the NE for the power competing game.

3.2 Convex Optimization Theory

Consider the optimization problem in the standard form:

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, ..., m \\ & h_j(x) = 0, j = 1, ..., p \end{array} \tag{3.1}$$

The domain is $\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom} f_i \cap \bigcap_{j=1}^{p} \mathbf{dom} h_i$. The Lagrangian is

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x) \tag{3.2}$$

where $\lambda = (\lambda_1, ..., \lambda_m)$ and $\nu = (\nu_1, ..., \nu_p)$ are the Lagrangian multipliers. The dual function is defined as:

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} (f_0(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)) \tag{3.3}$$

Thus the Lagrange dual problem is [31]:

$$\begin{aligned} &\text{maximize} \quad g(\lambda, \nu) \\ &\text{subject to} \quad \lambda_i \geq 0, i = 1, ..., m \end{aligned} \tag{3.4}$$

The dual function (3.3) yields lower bounds on the optimal value p^* of the problem (3.1) and we have $g(\lambda, \nu) \leq p^*$. The optimal value of the Lagrange dual problem (3.4) d^* is the best (largest) lower bound on p^* and we have $d^* \leq p^*$.

Theorem 3.2. *[31] (Slater's Condition) Strong duality ($d^* = p^*$) holds when the primal problems $f_0, ..., f_m$ of problem* (3.1) *are convex and there exists an $x \in$ **relint**$\mathcal{D}$ such that $f_i(x) < 0, i = 1, ..., m$ and $h_j(x) = 0, j = 1, ..., p$.*

□

For problem (3.1), when strong duality holds, x is optimal if and only if there exist λ, ν that satisfy Karush-Kuhn-Tucker (KKT) conditions [31]. The Lagrange, Slater's condition and KKT conditions was used when discussing the contraction property of the waterfilling operator.

3.3 Fixed Point Theory

Definition 3.1. [2] (Contraction Mapping) Many iterative algorithms can be written as

$$x(n+1) = T(x(n)), \quad n = 0, 1, ... \tag{3.5}$$

where T is a mapping from a subset $\mathcal{X}$ of $\mathcal{R}^n$ into itself and has the property

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|, \quad x, y \in \mathcal{X}. \tag{3.6}$$

$\|.\|$ is some norm and $\alpha \in [0, 1)$ is a constant. Such a mapping is called a contraction mapping and iteration (3.5) is called a contracting iteration. Scalar α is called the modulus of mapping T.

□

Definition 3.2. [2] (Fixed Point) For the mapping $T : \mathcal{X} \mapsto \mathcal{X}$, any vector $x^* \in \mathcal{X}$ satisfying $T(x^*) = x^*$ is called a fixed point of T and the iteration $x := T(x)$ can be viewed as an algorithm for finding such a fixed point.

□

Theorem 3.3. *[2, 32] (Existence and Uniqueness of the Fixed Point) Suppose that $T : \mathcal{X} \mapsto \mathcal{X}$ is a continuous and contraction mapping with modulus $\alpha \in [0,1)$; and that $\mathcal{X} \subseteq \mathcal{R}^n$ is nonempty, convex and compact (closed and bounded). Then there exists some unique x^* such that $x^* = T(x^*)$.*

□

The contraction property and theorem 3.3 was used for deriving conditions guaranteeing existence and uniqueness of the NE for the strategic power competing game.

3.4 Distributed Algorithm Theory

Nonlinear fixed-point problems are typically solved by iterative methods, especially distributed algorithms. From Definition 3.2, the iteration $x := T(x)$ can be viewed as an algorithm for finding such a fixed point. Three major schemes of the distributed algorithm have been considered for the power competing game according to the kinds of updating schedule [2]: Gauss-Seidel scheme, Jacobi scheme and Totally Asynchronous scheme. In this report, I propose a distributed algorithm based on Partially Asynchronous scheme.

Definition 3.3. [2] (Partially Asynchronous Algorithm Model) Let $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_Q$ be the given sets and let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times ... \times \mathcal{X}_Q$ be their Cartesian product. The mappings $T_q : \mathcal{X} \mapsto \mathcal{X}_q, q = 1, ..., Q$ and $T : \mathcal{X} \mapsto \mathcal{X}$ are given, where $T(x) = (T_1(x), ..., T_Q(x))$ and it is assumed to admit a fixed point $x^* = T(x^*)$. We consider the following distributed partially asynchronous iterative algorithm:

$$x_q^{(n+1)} = \begin{cases} T_q(x_1^{(\tau_1^q(n))}, ..., x_Q^{(\tau_Q^q(n))}), & \text{if} \quad n \in \mathcal{T}_q \\ x_q^{(n)}, & \text{otherwise} \end{cases}$$
$$\forall q \in \Omega = \{1, ..., Q\} \tag{3.7}$$

with $\tau_r^q(n)$ denoting the most recent time at which the interference from user r is perceived by user q at the n-th iteration and $\mathcal{T}_q$ denoting the set of times n at which $x_q^{(n)}$ is updated. $\tau_r^q(n)$ and $\mathcal{T}_q$ satisfy the following assumptions.

Assumption 3.1. (Partial Asynchronism) There exists a positive integer B such that:

a) For every q and for every $n \geq 0$, at least one of the elements of the set $\{n, n+1, ..., n+B-1\}$ belongs to $\mathcal{T}_q$.

b) There holds
$$n - B < \tau_r^q(n) \leq n, \tag{3.8}$$
for all r and q, and all $n \geq 0$ belonging to $\mathcal{T}_q$.

c) There holds $\tau_q^q(n) = n$ for all q and $n \in \mathcal{T}_q$.

□

We study the case where T has the following properties:

Assumption 3.2. By using the notations in Definition 3.3, it is assumed that:

a) The set $\mathcal{X}^*$ is nonempty and convex.

b) The function T is continuous and nonexpansive, that is, it satisfies

$$\|T(x) - x^*\|_\infty \leq \|x - x^*\|_\infty, \forall x \in \mathcal{X}, \forall x^* \in \mathcal{X}^* \tag{3.9}$$

c) For every $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ such that $\|x - x^*\|_\infty > 0$, there exists some i such that $\|x_i - x_i^*\| = \|x - x^*\|_\infty$ and $T_i(x) \neq x_i$.

d) If $x \in \mathcal{X}$, $T_i(x) \neq x_i$, and $x^* \in \mathcal{X}^*$, then $\|T_i(x) - x_i\| < \|x - x^*\|_\infty$.

□

We can see that Assumption 3.2(c) and 3.2(d) are closely related to the contraction property of the function T.

We have the following theorem showing that Assumption 3.2(d) is automatically true for certain algorithms involving a relaxation parameter.

Theorem 3.4. *[2] Suppose that a function $T : \mathcal{X} \mapsto \mathcal{X}$ satisfies Assumption 3.2(a), 3.2(b) and 3.2(c). Let $\gamma \in (0, 1)$. Then the mapping $F : \mathcal{X} \mapsto \mathcal{X}$ defined by*

$$F(x) = \gamma x + (1 - \gamma)T(x) \tag{3.10}$$

satisfies Assumption 3.2.

□

Then we have the following result:

Theorem 3.5. *[2] Suppose that $T : \mathcal{X} \mapsto \mathcal{X}$ satisfies Assumption 3.2 and suppose that Assumption 3.1 (Partial Asynchronism) holds. Then the sequence $\{x(n)\}$ generated by the asynchronous iteration $x := T(x)$ converges to some element of $\mathcal{X}^*$.*

□

We will utilize Theorem 3.5 to derive the conditions guaranteeing the convergence of the iterative distributed algorithm based on the partially asynchronous scheme.

3.5 Norm Representation

The contraction property of the mapping is norm-dependent. The choice of the proper norm is a critical issue and several norms are introduced in this section.

Definition 3.4. [2] (Block-maximum Norm) Given the mapping T from Definition 3.3 and $w = [w_1, ..., w_Q]^T > 0$, let $\|.\|_{F,blcok}^w$ denote the block-maximum norm, defined as

$$\|T(x)\|_{F,block}^w = \max_{q \in \Omega} \frac{\|T_q(x)\|_F}{w_q} \tag{3.11}$$

$\square$

Definition 3.5. [29] (Vector Weighted Maximum Norm) Given $w = [w_1, ..., w_Q]^T > 0$ and $x \in \mathcal{R}^Q$, let $\|.\|^w_{\infty,vec}$ denote the vector weighted maximum norm, defined as

$$\|x\|^w_{\infty,vec} = \max_{q\in\Omega} \frac{|x_q|}{w_q} \tag{3.12}$$

$\square$

Definition 3.6. [29] (Matrix Norm) Given $w = [w_1, ..., w_Q]^T > 0$ and $A \in \mathcal{R}^{Q\times Q}$, let $\|.\|^w_{\infty,mat}$ denote the matrix norm, defined as

$$\|A\|^w_{\infty,mat} = \max_{q\in\Omega} \frac{1}{w_q} \sum_{r=1}^{Q} |[A]_{qr}| w_r \tag{3.13}$$

$\square$

3.6 Others

Some other mathematic results which will be used are listed here.

Theorem 3.6. *[2] (Projection Theorem) For every $X \in \mathcal{C}^{n\times n}$, there exists a unique $Z \in \mathcal{X}$ that minimizes $\|Z - X\|_F$ over all $Z \in \mathcal{X}$, and will be denoted by $[X]^+$. The mapping $F : \mathcal{C}^{n\times n} \mapsto \mathcal{X}$ defined by $F(X) = [X]^+$ is continuous and nonexpensive, that is,*

$$\|[X]^+ - [Y]^+\|_F \leq \|X - Y\|_F, \quad \forall X, Y \in \mathcal{C}^{n\times n} \tag{3.14}$$

$\square$

Theorem 3.7. *[2] If M is a square nonnegative matrix, then the following are equivalent:*

1. $\rho(M) < 1$
2. *There exists some $w > 0$ such that $\|M\|^w_\infty < 1$*

$\square$

Chapter 4

Contraction Property of the Waterfilling Operator

From the previous chapter, we can see that the contraction property of the function is the key to the derivation of the conditions guaranteeing the existence and uniqueness of the NE and the convergence of the iterative distributed algorithm. This chapter will discuss the contraction property of the multi-user waterfilling operator, which is based on the interpretation of MIMO waterfilling operator as a matrix projection onto the convex set of feasible strategies of the users. The derivation of the contraction property is based on [27] and it will be outlined in this chapter.

4.1 Projection Interpretation

This section interprets the MIMO waterfilling operator as a matrix projection onto the convex set of feasible strategies of the users.

From equation (2.3), we have

$$R_q(Q_q, Q_{-q}) = \log\det(I + H_{qq}^H R_{-q}^{-1}(Q_{-q}) H_{qq} Q_q) = \log\det(I + R_{-q}^{-1}(Q_{-q}) H_{qq} Q_q H_{qq}^H) \quad (4.1)$$

and maximizing $\log\det(I + R_{-q}^{-1}(Q_{-q}) H_{qq} Q_q H_{qq}^H)$ is the same as maximizing $\log\det(R_{-q}(Q_{-q}) + H_{qq} Q_q H_{qq}^H)$. So game $\mathcal{G}$ can be equivalently written as:

$$\begin{array}{lrl} (\mathcal{G}') & \text{maximize}_{\{Q_q\}} & \log\det(R_{-q}(Q_{-q}) + H_{qq} Q_q H_{qq}^H) \\ & \text{subject to} & Q_q \in \mathcal{Q}_q, \quad \forall q \in \Omega \end{array} \quad (4.2)$$

It can be further simplified as $\forall q \in \Omega$

$$\begin{array}{lrl} (\mathcal{P}1) & \text{maximize}_{\{Q_q \succeq 0\}} & \log\det(R_{-q}(Q_{-q}) + H_{qq} Q_q H_{qq}^H) \\ & \text{subject to} & Tr\{Q_q\} \leq P_q \end{array} \quad (4.3)$$

which is actually a convex optimization problem.

Theorem 4.1. *[27] (Projection Interpretation of Waterfilling Mapping) The convex optimization problem $\mathcal{P}1$ has the same unique solution as the following convex optimization problem:*

$$\begin{aligned} (\mathcal{P}2) \qquad minimize_{\{Q_q \succeq 0\}} \quad & \frac{1}{2}\|Q_q - Q_0\|_F^2 \\ subject\ to \quad & Tr\{Q_q\} = P_q \end{aligned} \tag{4.4}$$

where $Q_0 = -(H_{qq}^H R_{-q}^{-1}(Q_{-q}) H_{qq})^{-1}$.

Proof. See Appendix A. □

By denoting the matrix projection of Q_0 with respect to the Frobeius norm onto the set $\mathcal{Q}_q$ by $[Q_0]_{\mathcal{Q}_q}$ and using Theorem 4.1, we have

Lemma 4.1. *The waterfilling operator $WF_q(Q_{-q})$ in (2.11) can be equivalently written as*

$$WF_q(Q_{-q}) = [-(H_{qq}^H R_{-q}^{-1}(Q_{-q}) H_{qq})^{-1}]_{\mathcal{Q}_q} \tag{4.5}$$

where $\mathcal{Q}_q$ is defined in (2.6). Thus from (2.12) the NE of the Game $\mathcal{G}$ can be obtained as the fixed-points of the mapping defined in (4.5):

$$Q_q^* = WF_q(Q_{-q}^*) = [-(H_{qq}^H R_{-q}^{-1}(Q_{-q}^*) H_{qq})^{-1}]_{\mathcal{Q}_q} \tag{4.6}$$

□

By Theorem 3.6 and the interpretation of the MIMO waterfilling as a projector, we have the following nonexpansive property of the waterfilling operator, which will be used to derive the contraction property of the MIMO waterfilling mapping.

Lemma 4.2. *Matrix projection $[.]_{\mathcal{Q}_q}$ is continuous and satisfies the nonexpansive property:*

$$\|[X]_{\mathcal{Q}_q} - [Y]_{\mathcal{Q}_q}\|_F \leq \|X - Y\|_F, \quad \forall X, Y \in \mathcal{C}^{n_{T_q} \times n_{T_q}} \tag{4.7}$$

□

4.2 Contraction Property

Define the nonnegative matrix $S \in \mathcal{R}_+^{Q \times Q}$ as

$$[S]_{qr} = \begin{cases} \rho(H_{rq}^H H_{qq}^{-H} H_{qq}^{-1} H_{rq}), & \text{if} \quad r \neq q \\ 0, & \text{otherwise} \end{cases} \tag{4.8}$$

where $\rho(A)$ denotes the spectral radius of matrix A.

By Definition 3.4, we define the block-maximum norm for waterfilling mapping as

$$\|WF(Q)\|_{F,block}^w = \max_{q \in \Omega} \frac{\|WF_q(Q_{-q})\|_F}{w_q} \tag{4.9}$$

where $WF(Q) = (WF_q(Q_{-q}))_{q\in\Omega} : \mathcal{Q} \mapsto \mathcal{Q}$. Therefore, we have the fixed-points mapping from (4.6):

$$Q^* = WF(Q^*) \tag{4.10}$$

Having the norm representation in Section 3.5, we have the following theorem on the contraction property of the waterfilling property.

Theorem 4.2. *[27] (Contraction Property of Waterfilling Mapping) Given* $w = [w_1, ..., w_Q]^T > 0$*, the mapping WF defined in* (4.9) *is Lipschitz continuous on* $\mathcal{Q}$*:*

$$\|WF(Q^{(1)}) - WF(Q^{(2)})\|^w_{F,block} \leq \|S\|^w_{\infty,mat}\|Q^{(1)} - Q^{(2)}\|^w_{F,block}, \quad \forall Q^{(1)}, Q^{(2)} \in \mathcal{Q} \tag{4.11}$$

Furthermore, if

$$\|S\|^w_{\infty,mat} < 1 \tag{4.12}$$

for some $w > 0$*, then mapping* WF *is a block-contraction with modulus* $\beta = \|S\|^w_{\infty,mat}$*.*

Proof. See Appendix B. □

Chapter 5

Existence and Uniqueness of the NE

Using the mathematical tools and the contraction property of the waterfilling operator, conditions for the existence and uniqueness of the NE can be derived.

5.1 Existence and Uniqueness of the NE

Theorem 5.1. *[27] (Existence of the NE) Game $\mathcal{G}$ always admits a NE.*

Proof. See Appendix C.1. □

Theorem 5.2. *[27] (Uniqueness of the NE) The NE of Game $\mathcal{G}$ is unique if*

$$\rho(S) < 1 \tag{5.1}$$

where S is defined in (4.8).

Proof. See Appendix C.2. □

5.2 Physical Interpretation

The condition $\rho(S) < 1$ indicates that $(I - S)$ is diagonally dominant, which means that the sum of each link's interference-to-signal ratio is less than one. This says that the uniqueness of a NE is guaranteed if the interference among the links is sufficiently small. The condition $\rho(S) < 1$ specifies the quantity of the interference to ensure the NE's uniqueness.

Chapter 6

MIMO Partially Asynchronous Iterative Waterfilling Algorithm

According to the discussion in Section 3.4, partially asynchronous algorithm model can be used to solve the fixed-point problem and reach the NE of game $\mathcal{G}$. According to this model, users are allowed to update their strategies asynchronously without waiting for other users. Thus some users may update their strategies more frequently. This gives more freedom to users than the synchronous ones, like Gauss-Seidel and Jacobi schemes. The term "partially" here corresponds to a time constraint on the information from other users. Totally asynchronous scheme allows users to update a possibly outdated information from other users while partially asynchronous scheme adds the time constraint and guarantees that other users' information is not outdated. Based on the waterfilling operator (2.11), partially asynchronous Iterative Waterfilling Algorithm (IWFA) will be discussed in this chapter.

6.1 Algorithm Description

We define the following notations for the formal description of the proposed partially asynchronous IWFA:

1. $\mathcal{T} = \{0, 1, 2, ...\}$ is the set of times.
2. $Q_q^{(n)}$ is the covariance matrix of the vector signal transmitted by user q at the n-th iteration.
3. $\mathcal{T}_q \in \mathcal{T}$ is the set of times n when $Q_q^{(n)}$ is updated.
4. $\tau_r^q(n)$ is the most recent time when the interference from user r is perceived by user q at the n-th iteration.
5. $Q_{-q}^{(\tau^q(n))} = (Q_1^{(\tau_1^q(n))}, ..., Q_{q-1}^{(\tau_{q-1}^q(n))}, Q_{q+1}^{(\tau_{q+1}^q(n))}, ..., Q_Q^{(\tau_Q^q(n))})$

For partially asynchronous IWFA, $\mathcal{T}_q$ and $\tau_r^q(n)$ should satisfy Assumption 3.1 (Partial Asynchronism). Using above notations and assumptions, the partially asynchronous IWFA is described in Algorithm 1.

Algorithm 1: MIMO Partially Asynchronous IWFA

Set $n = 0$ and $Q_q^{(0)}$ = any feasible covariance matrix;
for $n = 0$: Nit

$$Q_q^{(n+1)} = \begin{cases} WF_q(Q_{-q}^{(\tau^q(n))}), & \text{if} \quad n \in \mathcal{T}_q \\ Q_q^{(n)}, & \text{otherwise} \end{cases} \tag{6.1}$$
$$\forall q \in \Omega = \{1, ..., Q\}$$

end

This partially asynchronous IWFA can be implemented in a distributed way. When each user maximize his own information rate, he only need to measure the covariance matrix of the overall interference-plus-noise and then waterfill over this matrix.

Algorithm 1 is the generalization of sequential and simultaneous IWFAs [16], where the users update their own strategies sequentially and simultaneously. It relaxes the constraints on the synchronization of the users' updates.

Algorithm 1 is also a generalization of MIMO totally asynchronous IWFA [27], where the users updates their strategies based on possibly outdated information. When the time constraint B in Assumption 3.1 goes to infinity (i.e. no time constraint), partially asynchronous IWFA becomes totally asynchronous IWFA. This time constraint B avoid the users to use outdated information perceived from other users. The value of the time constraint B is up to the system design requirements.

We can generalize the partially asynchronous IWFA given in Algorithm 1 by introducing a memory in the updating process, as given in Algorithm 2. It is called smoothed partially asynchronous IWFA.

Algorithm 2: MIMO Smoothed Partially Asynchronous IWFA

Set $n = 0$ and $Q_q^{(0)}$ = any feasible covariance matrix and $\gamma_q \in [0, 1), \forall q \in \Omega = \{1, ..., Q\}$;
for $n = 0$: Nit

$$Q_q^{(n+1)} = \begin{cases} \gamma_q Q_q^{(n)} + (1 - \gamma_q) WF_q(Q_{-q}^{(\tau^q(n))}), & \text{if} \quad n \in \mathcal{T}_q \\ Q_q^{(n)}, & \text{otherwise} \end{cases} \tag{6.2}$$
$$\forall q \in \Omega$$

end

The factor $\gamma_q \in [0, 1)$ can be interpreted as a memory factor: the larger the γ_q is, the longer the memory of the algorithm is.

6.2 Convergence of the Algorithms

The convergence of the Algorithm 1 and 2 is guaranteed under the following sufficient conditions.

Theorem 6.1. *(Convergence of the Algorithms) Assume that the following condition is satisfied*

$$\rho(S) < 1 \tag{6.3}$$

where S is defined in (4.8) *and $\rho(S)$ is the spectral radius of S. Then, as $Nit \to \infty$, the MIMO Partially Asynchronous IWFA described in Algorithm* 1 *and the MIMO Smoothed Partially Asynchronous IWFA described in Algorithm* 2 *converge to the unique NE of game $\mathcal{G}$, for any set of feasible initial conditions and updating schedule.*

Proof. See Appendix D. □

We can see that the condition guaranteeing the convergence of the MIMO Partially Asynchronous IWFA coincides with the conditions ensuring the uniqueness of the NE of game $\mathcal{G}$; and it is independent on $\{\mathcal{T}_q\}$ and $\tau_r^q(n)$. Thus a unified condition is obtained.

Chapter 7 gives several simulations and will show that the convergence speed of Partially Asynchronous IWFA is in between of Simultaneous IWFA and Totaly Asynchronous IWFA.

Chapter 7

Numerical Analysis

In this chapter, some numerical simulation results will be provided together with some comparisons among proposed partially asynchronous IWFA and existing approaches. Eight Matlab files are writen to do the numerical simulations:

1. FYT.m: It is the main function for the simulation of the FYT project.
2. Random_Init.m: It randomly initializes the links, channel information and noise, which will be needed later. It will be called by FYT.m.
3. MIMO_Sequ_IWFA.m: It is the algorithm simulation - MIMO Sequential IWFA. It will be called by FYT.m.
4. MIMO_Simu_IWFA.m: It is the algorithm simulation - MIMO Simultaneous IWFA. It will be called by FYT.m.
5. MIMO_Part_Asyn_IWFA.m: It is the algorithm simulation - MIMO Partially Asynchronous IWFA. It will be called by FYT.m.
6. R_q.m: It calculates the information rate on link q. It will be called by MIMO_Sequ_IWFA.m, MIMO_Simu_IWFA.m and MIMO_Part_Asyn_IWFA.m.
7. WF_q.m: It is the waterfilling operator updates link q's power allocation according to other links' interference and the noise. It will be called by MIMO_Sequ_IWFA.m, MIMO_Simu_IWFA.m and MIMO_Part_Asyn_IWFA.m.
8. Rate_neg_q.m: It calculates the MUI plus noise covariance matrix observed by link q. It will be called by R_q.m and WF_q.m.

Figure 7.1 illustrates the internal relations of those eight Matlab simulation files and how they work. The arrow $A \rightarrow B$ indicates that file A is called by file B. The Matlab files will be included in the CD along with this report.

All the simulations included below considers the MIMO system having 4 links and each link has a 4×4 transmit-and-receive dimension. The number of links and the transmit-and-receive dimension for each link can be adjusted in file Random_Init.m. There are four numerical simulations.

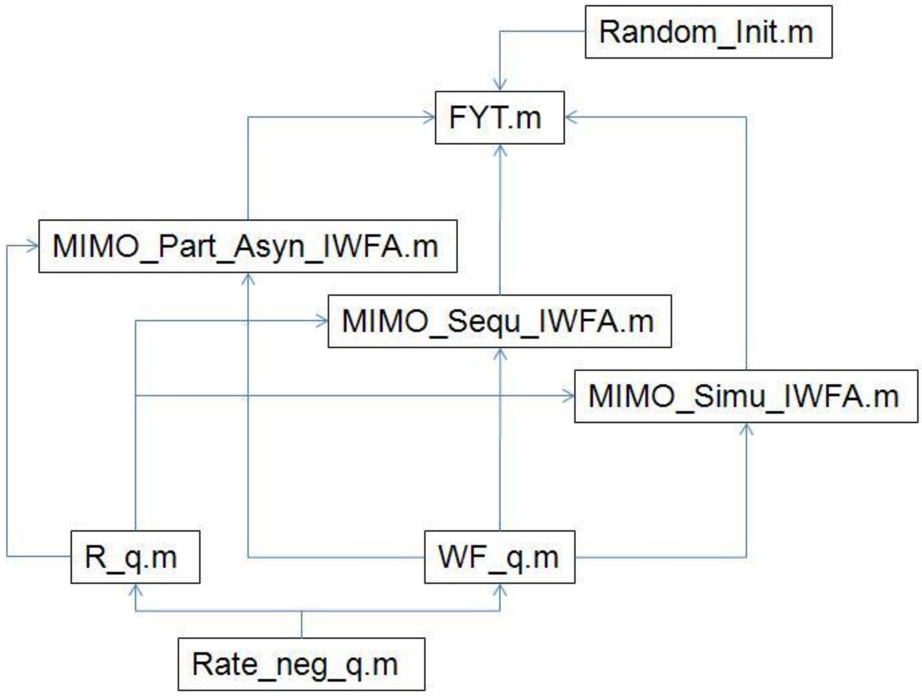

Figure 7.1: Relations of the Matlab Simulation Files

Simulation 1 - Sequential v.s. Simultaneous IWFA: Sequential and simultaneous IWFA have been proposed in some related work [16]. Figure 7.2 compares the rate evolution of the 4 links of those two algorithms. We can see that simultaneous IWFA is faster than sequential IWFA as expected as the users do not need to wait for other users scheduled in advance before updating his own power allocation.

Simulation 2 - Simultaneous v.s. Partially Asynchronous IWFA: Figure 7.3 compares the rate evolution of the 4 links of simultaneous and partially asynchronous IWFA. We can see that when the time constraint $B = 1$ for partially asynchronous IWFA, its rate evolution curves are the same as the simultaneous IWFA, which indicates that simultaneous IWFA is a special case of partially asynchronous IWFA. Actually, we can easily see this relation from Definition 3.3 and Assumption 3.1 that when the time constraint $B = 1$, each user needs to update their power strategy at each time, which is also the case for simultaneous IWFA.

Simulation 3 - Different Time Constraints for Partially Asynchronous IWFA: Figure 7.4 compares partially asynchronous IWFA with different time constraint B. We can see that the larger the value of B is, the longer the time it takes to converge. This makes sense as when B is larger, users may use outdated information and some user may updates his power strategy slower than other ones, which slows down the convergence speed. When B decreases to 1, partially asynchronous IWFA becomes simultaneous IWFA and the convergence speed is the fastest.

Simulation 4 - Pseudo-Totally v.s. Partially Asynchronous IWFA: Figure 7.5 compares the rate evolution of pseudo-totally and partially asynchronous IWFA. The term

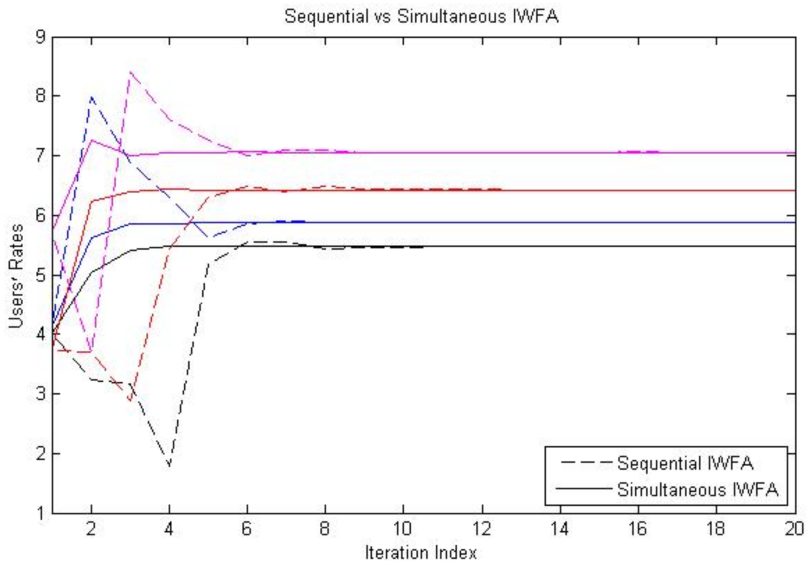

Figure 7.2: Sequential v.s. Simultaneous IWFA

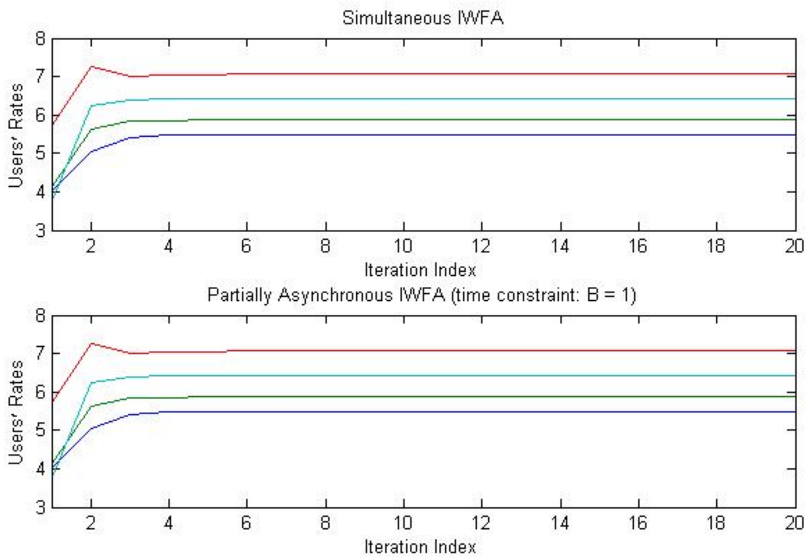

Figure 7.3: Simultaneous v.s. Partially Asynchronous IWFA

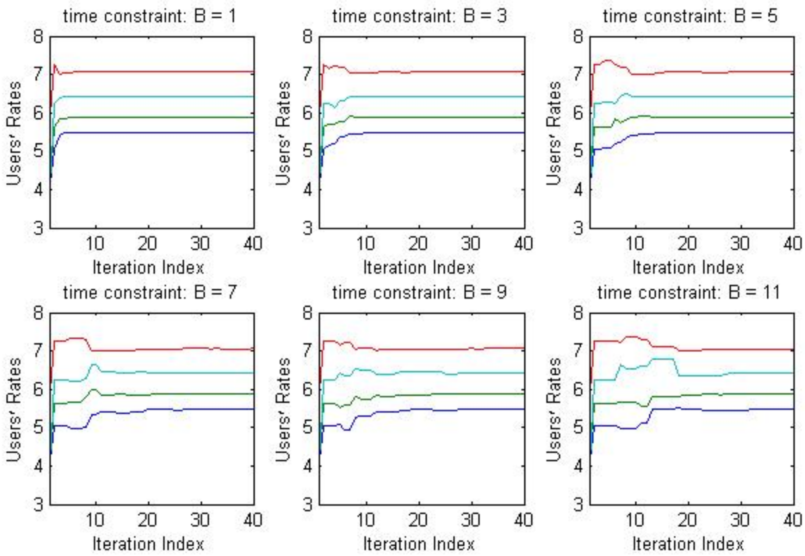

Figure 7.4: Different Time Constraints for Partially Asynchronous IWFA

pseudo here means that it is actually a partially asynchronous IWFA with a large value of time constraint B. If $B \to \infty$, which means no time constraint, partially asynchronous IWFA will become totally asynchronous IWFA. As we could not simulate infinite B, we use a time constraint value $B = 21$ to simulate totally asynchronous IWFA, thus *pseudo*. From this figure, we can easily see that totally asynchronous IWFA will be much slower than partially asynchronous IWFA with a reasonable value of time constraint B. Totally asynchronous IWFA provides users with the freedom of the choice of the outdated interference information and some user may update his power strategy slower, which slows down the convergence speed.

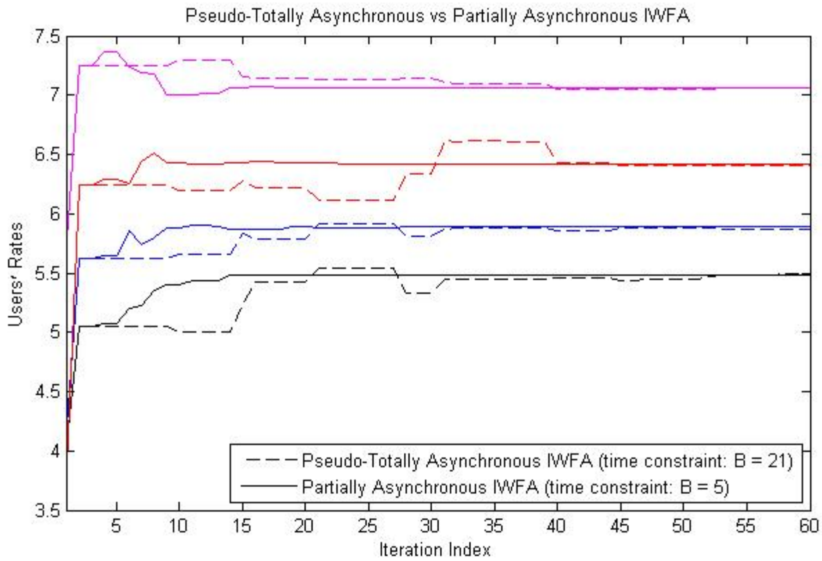

Figure 7.5: Pseudo-Totally v.s. Partially Asynchronous IWFA

Chapter 8

Conclusions and Future Work

This project considers the maximization of information rate on each link in the multiuser non-cooperative communication system given the constraints on the transmit power. In order to avoid the high complexity, heavy signaling and the coordination among the users required by a centralized solution, distributed ways are considered to solve the problem. By a game theoretical approach, this problem can be formulated as a strategic power competing game among users and iterative distributed algorithms can be used.

Firstly, I studied the mathematical tools required to solve the problem, for example, game theory, convex optimization and distributed algorithms. Then I did a survey on the related researches have been done and went through several important proofs derived by the authors. Finally, I proposed partially asynchronous Iterative Waterfilling Algorithm, namely partially asynchronous IWFA, to solve the problem. This algorithm is an extension on the time constraint requirement from the totally asynchronous IWFA and it is a generalization of sequential, simultaneous and totally asynchronous IWFAs. The main advantage of the proposed algorithm is that while no rigid scheduling in the updates of the users is required, the time constraint ensures the convergence speed of the algorithm: Users are allowed to choose their own strategies whenever they want within the time constraint and the convergence speed is ensured to certain level. The condition guaranteeing the convergence of the proposed algorithm has been derived, which is based on the contraction property of the waterfilling operator.

The proposed partially asynchronous IWFA is based on MIMO system with perfect channel state information (CSI). I originally considered the system with imperfect CSI; but I did not find a solution. The channel matrices are assumed to be square nonsingular. More general case on arbitrary channel matrices is discussed in [23]. Though the algorithm converges to the NE of the strategic power competing game, the NE is not Pareto-optimal[33]. It has been shown that Pareto-optimal solutions can be achieved as an NE of the game in selfish but cooperating systems [15, 17]. Further studies on those aspects will be valuable.

Appendix A

Proof of Theorem 4.1: Projection Interpretation of Waterfilling Mapping

Proof. Problem $\mathcal{P}1$ is a convex problem and satisfies Slater's condition. The Lagrangian function $\mathcal{L}$ for (4.3) is

$$\mathcal{L} = -\log\det(R_{-q}(Q_{-q}) + H_{qq}Q_qH_{qq}^H) - Tr(\Psi^T X) + \lambda(Tr(Q_q) - P_q) \tag{A.1}$$

Then strong duality holds for this problem and it leads to the KKT conditions (Theorem 3.2):

1. primal constraints:

$$-Q_q \preceq 0 \tag{A.2}$$
$$Tr(Q_q) - P_q \leq 0 \tag{A.3}$$

2. dual constraints:

$$\Psi \succeq 0 \tag{A.4}$$
$$\lambda \geq 0 \tag{A.5}$$

3. complementary slackness:

$$Tr(\Psi^H Q_q) = 0 \tag{A.6}$$
$$\lambda(Tr(Q_q) - P_q) = 0 \tag{A.7}$$

4. gradient of Lagrangian with respect to Q_q:

$$-H_{qq}^H(R_{-q}(Q_{-q}) + H_{qq}Q_qH_{qq}^H)^{-1}H_{qq} - \Psi + \lambda I = 0 \tag{A.8}$$

Firstly, let $Q_0 = -(H_{qq}^H R_{-q}^{-1}(Q_{-q})H_{qq})^{-1} \prec 0$, so we have

$$\begin{aligned} & H_{qq}^H(R_{-q}(Q_{-q}) + H_{qq}Q_qH_{qq}^H)^{-1}H_{qq} \\ = \; & (H_{qq}^{-H})^{-1}(R_{-q}(Q_{-q}) + H_{qq}Q_qH_{qq}^H)^{-1}(H_{qq}^{-1})^{-1} \\ = \; & (H_{qq}^{-1}R_{-q}(Q_{-q})H_{qq}^{-H} + Q_q)^{-1} \\ = \; & (Q_q + (H_{qq}^H R_{-q}^{-1}(Q_{-q})H_{qq})^{-1})^{-1} \\ = \; & (Q_q - Q_0)^{-1} \succ 0 \end{aligned} \tag{A.9}$$

Second, we know that $\lambda \neq 0$; otherwise, from (A.8) we have $0 \succ -H_{qq}^H(R_{-q}(Q_{-q}) + H_{qq}Q_qH_{qq}^H)^{-1}H_{qq} = \Psi \succ 0$. Thus $\lambda > 0$ and $Tr(Q_q) = P_q$ from (A.7).

Thirdly, $Tr(\Psi^H Q_q) = 0$ is equivalent to $\Psi^H Q_q = 0, \forall \Psi^H, Q_q \succeq 0$[34].

Then the KKT system for $\mathcal{P}1$ becomes:

$$Q_q \succeq 0 \tag{A.10}$$
$$Tr(Q_q) = P_q \tag{A.11}$$
$$\lambda > 0 \tag{A.12}$$
$$-(Q_q - Q_0)^{-1} + \lambda I \succeq 0 \tag{A.13}$$
$$\Psi^H Q_q = [-(Q_q - Q_0)^{-1} + \lambda I]^H Q_q = 0 \tag{A.14}$$

As $\lambda > 0$, let $\mu = -\frac{1}{\lambda}$, the KKT system for $\mathcal{P}1$ becomes:

$$Q_q \succeq 0 \tag{A.15}$$
$$Tr(Q_q) = P_q \tag{A.16}$$
$$\mu < 0 \tag{A.17}$$
$$Q_q - Q_0 + \mu I \succeq 0 \tag{A.18}$$
$$[(Q_q - Q_0) + \mu I]^H Q_q = 0 \tag{A.19}$$

Now let us look at problem $\mathcal{P}2$ which is also a convex problem and satisfies Slater's condition. The Lagrangian function $\mathcal{L}'$ for (4.4) is

$$\mathcal{L}' = \frac{1}{2}\|Q_q - Q_0\|_F^2 - Tr(\Gamma^H Q_q) + \nu(Tr(Q_q) - P_q) \tag{A.20}$$

The strong duality holds for this problem and it leads to the KKT conditions (Theorem 3.2):

1. primal constraints:

$$Q_q \succeq 0 \tag{A.21}$$
$$Tr(Q_q) = P_q \tag{A.22}$$

2. dual constraints:

$$\Gamma \succeq 0 \tag{A.23}$$
$$\nu \quad \text{is free} \tag{A.24}$$

3. complementary slackness:

$$Tr(\Gamma^H Q_q) = 0 \tag{A.25}$$
$$\nu(Tr(Q_q) - P_q) = 0 \tag{A.26}$$

4. gradient of Lagrangian with respect to Q_q:

$$(Q_q - Q_0) - \Gamma + \nu I = 0 \tag{A.27}$$

From (A.23) and (A.27), we have $Q_q - Q_0 + \nu I \succeq 0$. From (A.25) and (A.27), we have $[Q_q - Q_0 + \nu I]^H Q_q = 0$. Let $\nu = \mu$, we can easily see that the KKT system (A.15)-(A.19) for $\mathcal{P}1$ is equivalent to the KKT system for $\mathcal{P}2$ and it can represents the optimality conditions of problem $\mathcal{P}2$; which completes the proof. $\square$

Appendix B

Proof of Theorem 4.2: Contraction Property of Waterfilling Mapping

Given $Q^{(1)} = (Q_1^{(1)}, ..., Q_Q^{(1)}) \in \mathcal{Q}$ and $Q^{(2)} = (Q_1^{(2)}, ..., Q_Q^{(2)}) \in \mathcal{Q}$, define

$$\begin{aligned}
e_{WF_q} &= \quad \|WF_q(Q_{-q}^{(1)}) - WF_q(Q_{-q}^{(2)})\|_F, \quad \forall q \in \Omega && \text{(B.1)}\\
\mathbf{e}_{WF} &= \quad [e_{WF_1}, ..., e_{WF_Q}]^T && \text{(B.2)}\\
e_q &= \quad \|Q_q^{(1)} - Q_q^{(2)}\|_F, \quad \forall q \in \Omega && \text{(B.3)}\\
\mathbf{e} &= \quad [e_1, ..., e_Q]^T && \text{(B.4)}
\end{aligned}$$

Then we have

$$\begin{aligned}
e_{WF_q} &= \|WF_q(Q_{-q}^{(1)}) - WF_q(Q_{-q}^{(2)})\|_F && \text{(B.5)}\\
&= \|[-(H_{qq}^H R_{-q}^{-1}(Q_{-q}^{(1)})H_{qq})^{-1}]_{\mathcal{Q}_q} - [-(H_{qq}^H R_{-q}^{-1}(Q_{-q}^{(2)})H_{qq})^{-1}]_{\mathcal{Q}_q}\|_F && \text{(B.6)}\\
&\leq \|(H_{qq}^H R_{-q}^{-1}(Q_{-q}^{(1)})H_{qq})^{-1} - (H_{qq}^H R_{-q}^{-1}(Q_{-q}^{(2)})H_{qq})^{-1}\|_F && \text{(B.7)}\\
&\quad \text{(Lemma 4.2: Nonexpansive Property)} \\
&= \|H_{qq}^{-1}R_{-q}(Q_{-q}^{(1)})H_{qq}^{-H} - H_{qq}^{-1}R_{-q}(Q_{-q}^{(2)})H_{qq}^{-H}\|_F && \text{(B.8)}\\
&= \|[H_{qq}^{-1}R_{nq}H_{qq}^{-H} + H_{qq}^{-1}(\sum_{r\neq q} H_{rq}Q_r^{(1)}H_{rq}^H)H_{qq}^{-H}] \\
&\qquad - [H_{qq}^{-1}R_{nq}H_{qq}^{-H} + H_{qq}^{-1}(\sum_{r\neq q} H_{rq}Q_r^{(2)}H_{rq}^H)H_{qq}^{-H}]\|_F && \text{(B.9)}\\
&= \|H_{qq}^{-1}(\sum_{r\neq q} H_{rq}Q_r^{(1)}H_{rq}^H)H_{qq}^{-H} - H_{qq}^{-1}(\sum_{r\neq q} H_{rq}Q_r^{(2)}H_{rq}^H)H_{qq}^{-H}\|_F && \text{(B.10)}\\
&= \|H_{qq}^{-1}[\sum_{r\neq q} H_{rq}(Q_r^{(1)} - Q_r^{(2)})H_{rq}^H]H_{qq}^{-H}\|_F && \text{(B.11)}\\
&\leq \sum_{r\neq q} \|H_{qq}^{-1}H_{rq}(Q_r^{(1)} - Q_r^{(2)})H_{rq}^H H_{qq}^{-H}\|_F \quad \text{(Triangular Inequality)} && \text{(B.12)}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{r\neq q} \rho(H_{rq}^H H_{qq}^{-H} H_{qq}^{-1} H_{rq}) \|(Q_r^{(1)} - Q_r^{(2)})\|_F && \text{(B.13)}\\
&\quad (\|AXA^H\|_F \leq \lambda\text{max}(A^H A)\|X\|_F, \quad \text{where } X = X^H \text{ and } A \in \mathcal{C}^{n\times m} \text{ [29]})\\
&= \sum_{r\neq q} [S]_{qr} \|(Q_r^{(1)} - Q_r^{(2)})\|_F && \text{(B.14)}\\
&= \sum_{r\neq q} [S]_{qr} e_r && \text{(B.15)}
\end{aligned}$$

that is, $e_{WF_q} \leq \sum_{r\neq q} [S]_{qr} e_r$. By using (B.2) and (B.4), we have

$$\mathbf{0} \leq \mathbf{e}_{WF} \leq S\mathbf{e} \tag{B.16}$$

Apply weight maximum norm (Definition 3.5) on (B.16), we have

$$\|\mathbf{e}_{WF}\|_{\infty,vec}^w \leq \|S\mathbf{e}\|_{\infty,vec}^w \leq \|S\|_{\infty,mat}^w \|\mathbf{e}\|_{\infty,vec}^w, \quad \forall Q^{(1)}, Q^{(1)} \in \mathcal{Q}, w > 0 \tag{B.17}$$

Finally, by using (4.9) and (B.1)-(B.4), we have

$$\|WF(Q^{(1)}) - WF(Q^{(2)})\|_{F,block}^w \leq \|S\|_{\infty,mat}^w \|Q^{(1)} - Q^{(2)}\|_{F,block}^w \tag{B.18}$$

$\forall Q^{(1)}, Q^{(2)} \in \mathcal{Q}$ and $\forall w > 0$. If $\|S\|_{\infty,mat}^w < 1$, mapping WF is a block-contraction with modulus $\beta = \|S\|_{\infty,mat}^w$, which completes the proof.

Appendix C

Proof of Theorem 5.1 and 5.2

C.1 Proof of Theorem 5.1: Existence of the NE

1) $\mathcal{Q}_q$ defined in (2.6) is compact and convex. 2) R_q defined in (2.3) is continuous in $Q \in \mathcal{Q}$ and concave in $Q_q \in \mathcal{Q}_q$, which follows from the concavity of log function. Therefore, the existence of the NE can be proved by Theorem 3.1.

C.2 Proof of Theorem 5.2: Uniqueness of the NE

The uniqueness of the game $\mathcal{G}$ is guaranteed by the uniqueness of the solution of the fixed-point equation (4.6). We have 1) The joint strategy set $\mathcal{Q}$ is nonempty, convex and compact; 2) WF_q (4.5) is a continuous mapping from Lemma 4.1 and 4.2; and 3) WF_q is also a contraction mapping if $\|S\|^w_{\infty,mat} < 1$ from Theorem 4.2. Therefore, the existence of the NE can be proved by Theorem 3.3 if condition $\|S\|^w_{\infty,mat} < 1$ is satisfied. Since S is a square nonnegative matrix, the condition for the uniqueness of the NE can be $\rho(S) < 1$ from Theorem 3.7 for some $w > 0$, which proves the theorem. Note that Theorem 3.3 can also prove the existence of the NE, which is a second method different from the one presented in the proof of Theorem 5.1.

Appendix D

Proof of Theorem 6.1: Convergence of the Algorithms

Proof. We are going to utilize Theorem 3.5 to prove the convergence of the MIMO (Smoothed) Partially Asynchronous IWFA. Then the key questions are whether the waterfilling operator satisfies Assumption 3.2. Firstly, we define

$$\beta = \beta(w, S) = \|S\|_{\infty,mat}^{w} \tag{D.1}$$

where S is defined in (4.8). Since S is a square nonnegative matrix, $\rho(S) < 1$ is equivalent to $\|S\|_{\infty,mat}^{w} < 1$ from Theorem 3.7 for some $w > 0$. As the condition in Theorem 6.1 is $\rho(S) < 1$, thus condition $\beta = \|S\|_{\infty,mat}^{w} < 1$ is also satisfied. Therefore, contraction property of the waterfilling mapping is satisfied by Theorem 4.2:

$$\begin{aligned} \|WF(Q^{(1)}) - WF(Q^{(2)})\|_{F,block}^{w} &\leq \beta \|Q^{(1)} - Q^{(2)}\|_{F,block}^{w} \\ &< \|Q^{(1)} - Q^{(2)}\|_{F,block}^{w}, \quad \forall Q^{(1)}, Q^{(2)} \in \mathcal{Q} \end{aligned} \tag{D.2}$$

Secondly, we choose the block-maximum norm on $\mathcal{C}^{n\times n}$ (Section 3.5) in this report. We can easily see that the block-maximum norm $\|.\|_{F,block}^{w}$ corresponds to the notation $\|.\|_{\infty}$ used in Assumption 3.2. Thus $WF(Q)$ corresponds to $T(x)$ and $WF_i(Q_{-i})$ corresponds to $T_i(x)$, respectively.

Thirdly, we will go through all the assumptions in Assumption 3.2.

Assumption 3.2(a):
From Theorem 5.1 and 5.2, we know that there exists unique solution for game $\mathcal{G}$. Thus the solution set $\mathcal{Q}^*$ is nonempty and convex.

Assumption 3.2(b):
The function here is the waterfilling operator which has been proved to be continuous and nonexpansive by Lemma 4.1 and 4.2. Also, for $\forall Q \in \mathcal{Q}, Q^* \in \mathcal{Q}^*$, from (4.10) and (D.2) we have

$$\begin{aligned} \|WF(Q) - Q^*\|_{F,blcok}^{w} &= \|WF(Q) - WF(Q^*)\|_{F,blcok}^{w} \\ &< \|Q - Q^*\|_{F,blcok}^{w} \end{aligned} \tag{D.3}$$

Assumption 3.2(c):
Let $\forall Q \in \mathcal{Q}, Q^* \in \mathcal{Q}^*$ and $\|Q - Q^*\|_{F,blcok}^{w} > 0$. Assume that user $i \in \Omega = \{1, ..., Q\}$ satisfies

$$\|Q_i - Q_i^*\|_F^{w_i} = \max_i \frac{\|Q_i - Q_i^*\|_F}{w_i} = \|Q - Q^*\|_{F,blcok}^{w} \tag{D.4}$$

Therefore, from (4.10), (D.2) and (D.4), we have

$$\begin{aligned} \|WF_i(Q_{-i}) - Q_i^*\|_F^{w_i} &\leq \max_i \frac{\|WF_i(Q_{-i}) - Q_i^*\|_F}{w_i} \\ &= \|WF(Q) - Q^*\|_{F,blcok}^{w} \\ &= \|WF(Q) - WF(Q^*)\|_{F,blcok}^{w} \\ &< \|Q - Q^*\|_{F,blcok}^{w} \\ &= \|Q_i - Q_i^*\|_F^{w_i} \end{aligned} \tag{D.5}$$

Thus we get $WF_i(Q_{-i}) \neq Q_i$.

Assumption 3.2(d):
Let $Q \in \mathcal{Q}$, $WF_i(Q_{-i}) \neq Q_i$ and $Q^* \in \mathcal{Q}^*$, we have the following similar to (D.5):

$$\begin{aligned} \|WF_i(Q_{-i}) - Q_i^*\|_F^{w_i} &\leq \max_i \frac{\|WF_i(Q_{-i}) - Q_i^*\|_F}{w_i} \\ &= \|WF(Q) - Q^*\|_{F,blcok}^{w} \\ &= \|WF(Q) - WF(Q^*)\|_{F,blcok}^{w} \\ &< \|Q - Q^*\|_{F,blcok}^{w} \end{aligned} \tag{D.6}$$

By now, we have proved that the waterfilling operator satisfies all the assumptions in Assumption 3.2 and Assumption 3.1 (Partial Asynchronism) holds for MIMO Partially Asynchronous IWFA. Therefore, from Theorem 3.5, we know that the MIMO Partially Asynchronous IWFA described in Algorithm 1 converge to the unique NE of game $\mathcal{G}$.

As the waterfilling operator satisfies Assumption 3.2 (thus satisfies Assumption 3.2(a), (b) and (c)), from Theorem 3.4, we know that the smoothed waterfilling mapping

$$WF'(Q) = \gamma Q + (1 - \gamma) WF(Q), \quad \gamma \in (0, 1) \tag{D.7}$$

also satisfies Assumption 3.2. And when $\gamma = 0$, the smoothed waterfilling mapping becomes the original waterfilling mapping. Therefore, the smoothed waterfilling mapping actually satisfies Assumption 3.2 for $\gamma \in [0, 1)$. As Assumption 3.1 (Partial Asynchronism) holds for MIMO Smoothed Partially Asynchronous IWFA, from Theorem 3.5, we know that the MIMO Smoothed Partially Asynchronous IWFA described in Algorithm 2 also converge to the unique NE of game $\mathcal{G}$.

Finally, we complete the proof. □

Bibliography

[1] M. Osborne and A. Rubinstein, *A Course in Game Theory.* MIT press, 1994.

[2] D. Bertsekas and J. Tsitsiklis, *Parallel and Distributed Computation.* Old Tappan, NJ (USA); Prentice Hall Inc., 1989.

[3] R. Yates, "A framework for uplink power control in cellular radio systems," *IEEE Journal on Selected Areas in Communications*, vol. 13, no. 7, pp. 1341–1347, 1995.

[4] K. K. Leung, C. W. Sung, W. S. Wong, and T. M. Lok, "Convergence theorem for a general class of power-control algorithms," *IEEE transactions on communications*, vol. 52, no. 9, pp. 1566–1574, 2004.

[5] C. Sung and K. Leung, "A generalized framework for distributed power control in wireless networks," *IEEE Transactions on Information Theory*, vol. 51, no. 7, pp. 2625–2635, 2005.

[6] M. Xiao, N. Shroff, and E. Chong, "A utility-based power-control scheme in wireless cellular systems," *IEEE/ACM Transactions on Networking*, vol. 11, no. 2, pp. 210–221, 2003.

[7] H. Ji and C. Huang, "Non-cooperative uplink power control in cellular radio systems," *Wireless Networks*, vol. 4, no. 3, pp. 233–240, 1998.

[8] C. Saraydar, N. Mandayam, and D. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Transactions on Communications*, vol. 50, no. 2, pp. 291–303, 2002.

[9] M. Chiang, C. Tan, D. Palomar, D. ONeill, and D. Julian, "Power control by geometric programming," *Resource Allocation in Next Generation Wireless Networks*, p. 289, 2005.

[10] W. Yu, G. Ginis, and J. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE Journal on Selected Areas in Communications*, vol. 20, pp. 1105–1115, Jun 2002.

[11] S. T. Chung, S. J. Kim, J. Lee, and J. Cioffi, "A game-theoretic approach to power allocation in frequency-selective gaussian interference channels," *IEEE International Symposium on Information Theory, 2003. Proceedings.*, pp. 316–316, June-4 July 2003.

[12] G. Scutari, S. Barbarossa, and D. Ludovici, "On the maximum achievable rates in wireless meshed networks: centralized versus decentralized solutions," *IEEE International Conference on Acoustics, Speech, and Signal Processing, 2004. Proceedings. (ICASSP '04).*, vol. 4, pp. iv–573–iv–576 vol.4, May 2004.

[13] ——, "Cooperation diversity in multihop wireless networks using opportunistic driven multiple access," *4th IEEE Workshop on Signal Processing Advances in Wireless Communications, 2003. SPAWC 2003.*, pp. 170–174, June 2003.

[14] Z.-Q. Luo and J.-S. Pang, "Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines," *EURASIP J. Appl. Signal Process.*, vol. 2006, pp. 80–80, 2006.

[15] G. Scutari, D. Palomar, and S. Barbarossa, "Optimal Linear Precoding Strategies for Wideband Noncooperative Systems Based on Game Theory–Part I: Nash Equilibria," *IEEE Transactions on Signal Processing*, vol. 56, no. 3, p. 1230, 2008.

[16] ——, "Optimal Linear Precoding Strategies for Wideband Non-Cooperative Systems Based on Game Theory–Part II: Algorithms," *IEEE Transactions on Signal Processing*, vol. 56, no. 3, p. 1250, 2008.

[17] E. Larsson and E. Jorswieck, "The MISO interference channel: Competition versus collaboration," in *Allerton Conference on Communication, Control, and Computing*, 2007.

[18] S. Ye and R. Blum, "Optimized signaling for MIMO interference systems with feedback," *IEEE Transactions on Signal Processing*, vol. 51, no. 11, pp. 2839–2848, 2003.

[19] M. Demirkol and M. Ingram, "Power-controlled capacity for interfering MIMO links," in *IEEE VTS 54th Vehicular Technology Conference, 2001. VTC 2001 Fall*, vol. 1, 2001.

[20] C. Liang and K. Dandekar, "Power management in MIMO ad hoc networks: A game-theoretic approach," *IEEE Transactions on Wireless Communications*, vol. 6, no. 4, p. 1164, 2007.

[21] G. Arslan, M. Demirkol, and Y. Song, "Equilibrium efficiency improvement in MIMO interference systems: a decentralized stream control approach," *IEEE Transactions on Wireless Communications*, vol. 6, no. 8, pp. 2984–2993, 2007.

[22] G. Scutari, D. Palomar, and S. Barbarossa, "Competitive design of multiuser mimo systems based on game theory: A unified view," *IEEE Journal on Selected Areas in Communications*, vol. 26, pp. 1089–1103, September 2008.

[23] ——, "The mimo iterative waterfilling algorithm," *accepted in IEEE Trans. on Signal Processing*, submitted June 2008.

[24] K. Shum, K. Leung, and C. Sung, “Convergence of iterative waterfilling algorithm for Gaussian interference channels,” *IEEE Journal on Selected Areas in Communications*, vol. 25, no. 6, pp. 1091–1100, 2007.

[25] G. Scutari, D. Palomar, and S. Barbarossa, “Simultaneous iterative water-filling for Gaussian frequency-selective interference channels,” in *Proc. of the 2006 IEEE International Symposium on Information Theory (ISIT 2006)*, 2006.

[26] ——, “Distributed Totally Asynchronous Iterative Waterfilling for Wideband Interference Channel with Time/Frequency Offset,” in *Proc. of the IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP), Honolulu, Hawaii, USA*, 2007.

[27] ——, “Asynchronous iterative waterfilling for Gaussian frequency-selective interference channels: a unified framework,” in *Information Theory and Applications Workshop, 2007*, 2007, pp. 349–358.

[28] T. Cover, J. Thomas, J. Wiley, and W. InterScience, *Elements of information theory.* Wiley New York, 1991.

[29] R. Horn and C. Johnson, *Matrix analysis.* Cambridge university press, 1985.

[30] J. Rosen, “Existence and uniqueness of equilibrium points for concave n-person games,” *Econometrica*, vol. 33, pp. 520–534, 1965.

[31] S. Boyd and L. Vandenberghe, *Convex optimization.* Cambridge university press, 2004.

[32] M. M. R.P. Agarwal and D. O’Regan, *Fixed Point Theory and Applications.* Cambridge University Press, 2001.

[33] P. Dubey, “Inefficiency of Nash equilibria,” *Mathematics of Operations Research*, pp. 1–8, 1986.

[34] D. Bernstein, *Matrix mathematics: theory, facts, and formulas with application to linear systems theory.* Princeton University Press, 2005.